A Characterization of Graphs with Disjoint Total Dominating Sets

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Abstract

A set $S$ of vertices in a graph $G$ is a total dominating set of $G$ if every vertex is adjacent to a vertex in $S$. A fundamental problem in total domination theory in graphs is to determine which graphs have two disjoint total dominating sets. In this paper, we solve this problem and provide a constructive characterization of the graphs that have two disjoint total dominating sets.

Keywords: Total domination number; Disjoint total dominating sets.

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1 Introduction

A dominating set of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex not in $S$ has a neighbor in $S$, where two vertices are neighbors if they are adjacent. A total dominating set of a graph $G$ with no isolated vertex is a set $S$ of vertices such that every vertex in $G$ has a neighbor in $S$. Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes,

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Hedetniemi, and Slater [5, 6]. For a recent book on total domination in graphs we refer the reader to [12]. A survey of total domination in graphs can also be found in [8].

A classical result in domination theory, due to Ore [13] in 1962, is that every graph with no isolated vertex has two disjoint dominating sets. However, it is not the case that every graph with no isolated vertex can be partitioned into a dominating set and a total dominating set. Henning and Southey [10] showed that every connected graph with minimum degree at least two that is not a cycle on five vertices has a disjoint dominating set and a total dominating set. Further, in [11] they present a constructive characterization of connected graphs of order at least 4 that have a disjoint dominating set and a total dominating set. Disjoint dominating and total dominating sets in graphs are studied further, for example, in [9]. A characterization of graphs with disjoint dominating and paired-dominating sets is characterized in [14].

It remains, however, an outstanding problem to determine which graphs have two disjoint total dominating sets. Zelinka [15] in 1989 showed that no minimum degree condition in a graph is sufficient to guarantee that there exist two disjoint total dominating sets in the graph. Heggernes and Telle [7] showed that the decision problem to decide for a given graph \( G \) if it has two disjoint total dominating sets is NP-complete, even for bipartite graphs.

The total domatic number \( t_{\text{dom}}(G) \) of \( G \) is the maximum number of disjoint total dominating sets [2]. This can also be considered as a coloring of the vertices such that every vertex has a neighbor of every color (and has been called the coupon coloring problem [3]). Recent work on the total domatic number can be found, for example, [1, 4]. The fundamental problem in total domination theory in graphs of determining which graphs have two disjoint total dominating sets can be phrased as follows: Determine which graphs \( G \) satisfy \( t_{\text{dom}}(G) \geq 2 \). We call a graph a TDP-graph (standing for “total dominating partitionable graph”) if its vertex set can be partitioned into two total dominating sets; that is, a graph \( G \) is a TDP-graph if and only if \( t_{\text{dom}}(G) \geq 2 \).

In this paper, we provide a constructive characterization of the graphs that have two disjoint total dominating sets, or, equivalently, a characterization of the TDP-graphs. We describe a procedure to build TDP-graphs in terms of a labeling of the vertices that indicate the role each vertex plays in the sets associated with the two disjoint total dominating sets. We show that the resulting family we construct, starting from four initial base graphs and applying one of nineteen operations to extend graphs in the family to larger graphs, is precisely the class of all TDP-graphs.

### 1.1 Notation

For notation and graph theory terminology we generally follow [12]. The order of \( G \) is denoted by \( n(G) = |V(G)| \), and the size of \( G \) by \( m(G) = |E(G)| \). We denote the degree of a vertex \( v \) in the graph \( G \) by \( d_G(v) \). The maximum (minimum) degree among the vertices of \( G \) is denoted by \( \Delta(G) \) (\( \delta(G) \), respectively). The open neighborhood of \( v \) is \( N_G(v) = \{u \in V(G) | uv \in E(G)\} \). For a set \( S \subseteq V(G) \), its open neighborhood is the set \( N_G(S) = \bigcup_{v \in S} N_G(v) \). For subsets \( X \) and \( Y \) of vertices of \( G \), we denote the set of edges
with one end in \( X \) and the other end in \( Y \) by \([X,Y]\). For a set \( S \subseteq V(G) \), the subgraph induced by \( S \) is denoted by \( G[S] \). Further, the subgraph of \( G \) obtained from \( G \) by deleting all vertices in \( S \) and all edges incident with vertices in \( S \) is denoted by \( G - S \); that is, \( G - S = G[V(G) \setminus S] \). If \( S = \{v\} \), we simply denote \( G - \{v\} \) by \( G - v \).

The distance between two vertices \( u \) and \( v \) in \( G \), denoted \( d_G(u,v) \), is the minimum length of a \((u,v)\)-path in \( G \). By \( W_{uv} \) we denote the set of all vertices of \( G \) which are closer to \( u \) than to \( v \); that is, \( W_{uv} = \{w \mid d_G(w,u) < d_G(w,v)\} \). Symmetrical, \( W_{vu} \) is defined. A block of a graph \( G \) is a maximal connected subgraph of \( G \) which has no cut-vertex of its own. A block containing exactly one cut-vertex is called an end-block. It is well known that any two blocks of a graph have at most one vertex in common, namely a cut-vertex. Further, a connected graph with at least one cut-vertex has at least two end-blocks. Let \( X \) denote the set of cut-vertices of a connected graph \( G \) and let \( Y \) denote the set of its blocks. The block graph of \( G \) is a bipartite graph with partite sets \( X \) and \( Y \) in which a vertex \( x \in X \) is adjacent to a vertex \( y \in Y \) if \( x \) is a vertex of the block \( y \). It is well-known that the block graph of any connected graph is a tree.

Let \( u \) be a cut-vertex of a graph \( G \). Let \( H_1 \) and \( H_2 \) be two vertex disjoint subgraphs of \( G - u \) that contain all the components of \( G - u \), where each of \( H_1 \) and \( H_2 \) contain at least one component of \( G - u \). We call \( H_1 \) and \( H_2 \) the associated subgraphs of \( G - u \). For \( i \in [2] \), we denote by \( H_i^u \) the subgraph of \( G \) induced by \( V(H_i) \cup \{u\} \). Further, the vertex in \( H_1^u \) named \( u \) we rename \( u' \), and the vertex in \( H_2^u \) named \( u \) we rename \( u'' \) in order to distinguish between \( u, u' \) and \( u'' \). We use the standard notation \( [k] = \{1,2,\ldots,k\} \).

2 The Graph Family \( \mathcal{G} \)

In this section, we construct a graph family \( \mathcal{G} \) such that every graph in the family has two disjoint total dominating sets. First, we define a labeling of a graph \( G \) as a partition \( S = (S_A,S_B) \) of \( V(G) \). The label or status of a vertex \( v \), denoted \( \text{sta}(v) \), is the letter \( X \in \{A,B\} \) such that \( v \in S_X \). We denote by \( \overline{X} \) the set \( \{A,B\} \setminus X \). We denote by \((G,S)\) a graph \( G \) with a given labeling \( S \). Our aim is to describe a procedure to build TDP-graphs in terms of labelings. For \( i \in [4] \), by a labeled-\( G_i \), we shall mean the graph \( G_i \) and its associated labeling shown in Fig. 1. Further, we call each labeled-\( G_i \) a labeled base graph.

![Figure 1: The four labeled base graphs \( G_1, G_2, G_3, G_4 \).](image)

Let \( \mathcal{G} \) be the minimum family of labeled graphs that: (i) contains the four labeled base graphs; and (ii) is closed under the nineteen operations \( O_1 \) through to \( O_{19} \) listed below, which extend a labeled graph \((G',S')\) to a new labeled graph \((G,S)\). In Fig. 2-9, the vertices
of $G'$ are colored black and the new vertices of $G$ are colored white.

Operation $O_1$: $(G, S)$ is obtained from $(G', S')$ by adding an edge between two nonadjacent vertices of the same status. See the upper diagram of Fig. 2.

Operation $O_2$: $(G, S)$ is obtained from $(G', S')$ by adding an edge between two nonadjacent vertices of different status. See the lower diagram of Fig. 2.

Operation $O_3$: If $u$ and $v$ are distinct vertices of different status from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by adding a new vertex of any status adjacent to both $u$ and $v$. See the left diagram in the upper part of Fig. 3.

Operation $O_4$: If $u$ and $v$ are distinct vertices of the same status from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by adding adjacent vertices $x$ and $y$ and edges $ux$ and $vy$ with $\text{sta}(x) = \text{sta}(y) \neq \text{sta}(u)$. See the middle diagram in the upper part of Fig. 3.

Operation $O_5$: If $u$ and $v$ are distinct vertices of different status from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by adding adjacent vertices $x$ and $y$ and edges $ux$ and $vy$ with $\text{sta}(x) = \text{sta}(u)$. See the right diagram in the upper part of Fig. 3.

Operation $O_6$: If $u$ and $v$ are distinct vertices of the same status from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by adding a path $xyz$ with $\text{sta}(y) = \text{sta}(z) \neq \text{sta}(x) = \text{sta}(u)$ and adding edges $ux$ and $vy$. See the left diagram in the lower part of Fig. 3.

Operation $O_7$: If $u$ and $v$ are distinct vertices of the same status from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by adding a path $xyzw$ and edges $ux$ and $vw$ with $\text{sta}(x) = \text{sta}(w) = \text{sta}(u) \neq \text{sta}(y) = \text{sta}(z)$. See the middle diagram in the lower part of Fig. 3.

Figure 2: The operations $O_1$ and $O_2$.

Figure 3: The operations $O_3$-$O_8$. 

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Operation $O_8$: If $u$ and $v$ are distinct vertices of different status from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by adding a path $xyzw$ and edges $ux$ and $vw$ with $\text{sta}(x) = \text{sta}(y) = \text{sta}(v) \neq \text{sta}(z) = \text{sta}(w)$. See the right diagram in the lower part of Fig. 3.

Operation $O_9$: If $u$ and $v$ are adjacent vertices of different status from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by subdividing $uv$ with four consecutive vertices $x, y, z, w$ where $x$ is adjacent to $u$ and $\text{sta}(u) = \text{sta}(z) = \text{sta}(w) \neq \text{sta}(x) = \text{sta}(y)$. See the upper diagram of Fig. 4.

Operation $O_{10}$: If $u$ and $v$ are adjacent vertices of the same status from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by subdividing $uv$ with four consecutive vertices $x, y, z, w$ where $x$ is adjacent to $u$ and $\text{sta}(u) = \text{sta}(x) = \text{sta}(w) \neq \text{sta}(y) = \text{sta}(z)$. See the lower diagram of Fig. 4.

Figure 4: The operations $O_9$ and $O_{10}$.

Operation $O_{11}$: If $v$ is a vertex from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by adding an edge $xy$ together with the edges $vx$ and $vy$ where $\text{sta}(x) = \text{sta}(y) \neq \text{sta}(v)$. See the left diagram of Fig. 5.

Operation $O_{12}$: If $v$ is a vertex from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by adding a path $xyz$ together with the edges $vx$ and $vz$ where $\text{sta}(x) = \text{sta}(y) \neq \text{sta}(z) = \text{sta}(v)$. See the middle diagram of Fig. 5.

Operation $O_{13}$: If $v$ is a vertex from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by adding a path $xyzw$ together with the edges $vx$ and $vw$ where $\text{sta}(x) = \text{sta}(w) = \text{sta}(v) \neq \text{sta}(y) = \text{sta}(z)$. See the right diagram of Fig. 5.

Figure 5: The operations $O_{11}$, $O_{12}$ and $O_{13}$. 
Operation $O_{14}$: If $v$ is a vertex from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by adding a 3-cycle, $xyzx$, together with the edge $vx$ where $sta(x) = sta(v)$ $\neq sta(y) = sta(z)$. See the left diagram of Fig. 6.

Operation $O_{15}$: If $v$ is a vertex from $(G', S')$ of any status, then $(G, S)$ is obtained from $(G', S')$ by adding a 4-cycle, $xyzwx$, together with the edge $vx$ where $sta(x) = sta(y)$ $\neq sta(z) = sta(w)$. See the middle diagram of Fig. 6, where the notation $X/X$ means that the vertex can have any status.

Operation $O_{16}$: If $v$ is a vertex from $(G', S')$, then $(G, S)$ is obtained from $(G', S')$ by adding a 5-cycle, $xyzwtx$, together with the edge $vx$ where $sta(x) = sta(y) = sta(t) \neq sta(z) = sta(w) = sta(v)$. See the right diagram of Fig. 6.

[Diagram of Fig. 6 showing $O_{14}$, $O_{15}$, and $O_{16}$ operations]

Operation $O_{17}$: If $u$ is a cut-vertex from $(G', S')$ with associated subgraphs $H_1^u$ and $H_2^u$, and in $N_{H_1^u}(u')$ there exists a vertex of the same status as $u$ and in $N_{H_2^u}(u'')$ there exists a vertex of different status as $u$, then $(G, S)$ is obtained from $H_1^u$ and $H_2^u$ by adding a new vertex $v$ and the edges $u'v$ and $vu''$. The status of all vertices from $H_1^u$ remains the same as in $G'$, $sta(v) = sta(u'') \neq sta(u') = sta(u)$ and the status of all vertices from $H_2^u$ is exchanged with respect to their status in $G'$. See the diagram of Fig. 7, where the notation $\overline{A}$ means that the status of all vertices of $A$ from $G'$ was changed in $G$.

[Diagram of Fig. 7 showing $O_{17}$ operation]

Operation $O_{18}$: If $uv$ is a bridge from $(G', S')$ with $sta(u) = sta(v)$, then $(G, S)$ is obtained by subdividing the edge $uv$ twice with new vertices $x$ and $y$, where $x$ is adjacent to $u$ and where $sta(u) = sta(x) \neq sta(y)$, and where the status of all vertices from $W_{vu}$ in $G'$ are changed in $G$. See Fig. 8, where the notation $\overline{W_{vu}}$ means that the status of all vertices of $W_{vu}$ from $G'$ was changed in $G$.

[Diagram of Fig. 8 showing $O_{18}$ operation]
Operation $O_{19}$: If $uv$ is a bridge from $(G', S')$ with $\text{sta}(u) \neq \text{sta}(v)$, then $(G, S)$ is obtained by subdividing the edge $uv$ twice with new vertices $x$ and $y$, where $x$ is adjacent to $u$, and where $\text{sta}(x) = \text{sta}(y) \neq \text{sta}(u)$, and where the status of all vertices from $W_{vu}$ in $G'$ are changed in $G$. See Fig. 9, where $W_{vu}$ is as defined in Operation $O_{18}$.

We remark that, by definition, all operations $O_3$ to $O_{19}$ produce new vertices. Further, exactly one new vertex created in each of the operations $O_{14}$ to $O_{16}$ has degree 3, and all other new vertices created using operations $O_3$ to $O_{19}$ have degree 2 in $G$. Moreover all operations from $O_{11}$ to $O_{19}$ produce new cut vertices. In this sense all operations, except $O_1$ and $O_2$, can be viewed as base operations which build the sparse skeleton of TDP-graphs, while $O_1$ and $O_2$ fill this skeleton with additional edges. This is also the main idea of the proof. First to discard all edges which are there by one of the operations $O_1$ and $O_2$, and then study the resulting vertices of degree two.

**Lemma 1** If $(G, S) \in \mathcal{G}$ for some labeling $S = (S_A, S_B)$, then $G$ is a TDP-graph. Further, $S = (S_A, S_B)$ is a partition of $V(G)$ into two total dominating sets of $G$.

**Proof.** We proceed by induction on the number, $k \geq 0$, of operations $O_1$ through $O_{19}$ used to construct a labeled graph $(G, S) \in \mathcal{G}$. If $k = 0$, then $(G, S)$ is one of the four labeled base graphs illustrated in Fig. 1, and one can readily observe that $G$ is a TDP-graph and $S = (S_A, S_B)$ is a partition of $V(G)$ into two total dominating sets of $G$. This establishes the base case. Let $k \geq 1$ and suppose that every labeled graph $(G', S') \in \mathcal{G}$ that can be constructed using fewer than $k$ operations satisfies the desired result.

Let $(G, S) \in \mathcal{G}$ be a labeled graph that can be built from one of the labeled base graphs by a sequence of $k$ operations $O_1$-$O_{19}$. Let $O_j$ be the last operation of that sequence where $j \in [19]$, and let $(G', S')$ be the graph obtained from the same labeled base graph with the same sequence as that used to construct $(G, S)$ but without applying the last operation $O_j$. Thus, $(G', S') \in \mathcal{G}$ can be constructed using fewer than $k$ operations. By the induction hypothesis, the graph $G'$ is a TDP-graph and $S' = (S'_A, S'_B)$ is a partition of $V(G')$ into two total dominating sets of $G'$. If $j \in [2]$, then $S = S'$ and $G$ is a TDP-graph since no new vertices were added. For $3 \leq j \leq 19$ it is a simple exercise to check from the status of the new vertices added to $(G', S')$ when forming $(G, S)$ that the operation $O_j$ yields two disjoint total domination sets, namely $S_A$ and $S_B$. Thus, $G$ is a TDP-graph, and $S = (S_A, S_B)$ is a partition of $V(G)$ into two total dominating sets of $G$. $\square$
3 Main Result

Our main result is to provide a constructive characterization of the graphs that have two disjoint total dominating sets, or, equivalently, a characterization of the TDP-graphs. We prove that the class of all TDP-graphs is precisely the family $\mathcal{G}$ constructed in Section 2. A proof of Theorem 2 is given in Section 4.

Theorem 2 A graph $G$ is a TDP-graph if and only if every component of $(G, S)$ is in $\mathcal{G}$ for some labeling $S$. Further, if $(G, S) \in \mathcal{G}$, then $S = (S_A, S_B)$ is a partition of $V(G)$ into two total dominating sets of $G$.

4 Proof of Theorem 2

The sufficient follows from Lemma 1. To prove the necessity, let $G$ be a TDP-graph and let $S = (S_A, S_B)$ be a partition of $V(G)$ into two total dominating sets of $G$. We show that $(G, S) \in \mathcal{G}$ by induction on $m = |E(G)|$. Since $G$ is a TDP-graph, we note that $\delta(G) \geq 2$, $G$ has order $n \geq 4$, and $m \geq 4$. If $m = 4$, then necessarily $G \cong C_4$, and $(G, S)$ is the labeled base graph $G_1$, and so $(G, S) \in \mathcal{G}$. This establishes the base case. Let $m \geq 5$ and assume that every TDP-graph $G'$ of size less than $m$ where $S' = (S'_A, S'_B)$ is a partition of $V(G')$ into two total dominating sets satisfies $(G', S') \in \mathcal{G}$.

Let $G$ be a TDP-graph of order $n$ and size $m$, and let $S = (S_A, S_B)$ be a partition of $V(G)$ into two total dominating sets of $G$. If $G$ is disconnected, we apply the inductive hypothesis to each component of $G$ to produce the desired result. Hence, we may assume that $G$ is connected.

Our general strategy in what follows is to reduce the graph $G$ to a TDP-graph $G'$ of size less than $m$, apply the inductive hypothesis to $G'$ to show that $(G', S') \in \mathcal{G}$, and then reconstruct the graph $(G, S)$ from $(G', S')$ by applying one of the operations $O_x$, $x \in [19]$, to show that $(G, S) \in \mathcal{G}$. We state this formally, since we will frequently use the following statement.

Statement 1 If $G'$ is a TDP-graph of size less than $m$, where $S' = (S'_A, S'_B)$ is a partition of $V(G')$ into two total dominating sets, and $(G, S)$ can be constructed from $(G', S')$ by applying one of the operations $O_x$, where $x \in [19]$, then $(G, S) \in \mathcal{G}$.

We define three graphs $G_A$, $G_B$ and $G_{AB}$ associated with the graph $G$ and the partition $S = (S_A, S_B)$. Let $G_A$ and $G_B$ be the subgraphs of $G$ induced by the sets $S_A$ and $S_B$, respectively, and so $G_A = G[S_A]$ and $G_B = G[S_B]$. Let $G_{AB}$ be the (spanning) subgraph of $G$ with $V(G_{AB}) = V(G)$ and $E(G_{AB}) = E(G) \setminus (E(G_A) \cup E(G_B))$.

Claim 1 If some component of $G_A$, $G_B$ or $G_{AB}$ is not a star, then $(G, S) \in \mathcal{G}$.
Proof. Suppose that there exists a component, $C$, of $G_A$, $G_B$ or $G_{AB}$ which is not a star. If $C$ contains a cycle $v_1 \ldots v_kv_1$, $k \geq 3$, then $G$ can be obtained from $G' = G - v_1v_2$ by either applying operation $O_1$ in the case when $C$ is a component of $G_A$ or $G_B$ or by applying operation $O_2$ in the case when $C$ is a component of $G_{AB}$. If $C$ contains no cycle, then $C$ is a tree different from a star. Therefore, there exists a path $v_1v_2u_3u_4$ in $C$ and $G$ can be obtained from $G' = G - u_2u_3$ by either applying operation $O_1$ in the case when $C$ is a component of $G_A$ or $G_B$ or by applying operation $O_2$ in the case when $C$ is a component of $G_{AB}$. In all cases, since $S = (S_A, S_B)$ is a partition of $V(G)$ into two total dominating sets of $G$, the same partition $S' = S = (S_A, S_B)$ is a partition of $V(G')$ into two total dominating sets of $G'$. By the inductive hypothesis, $(G', S') \in \mathcal{G}$. We can obtain $G$ from the same labeled base graph as $G'$ and the same sequence of operations from $O_1$-$O_{19}$ used to construct $(G', S')$ by adding at the end of this sequence the operation $O_1$ or $O_2$. Hence $(G, S) \in \mathcal{G}$. (c)

By Claim 1, we may assume that every component of $G_A$, $G_B$ or $G_{AB}$ is a star, for otherwise the desired result follows. We call the resulting graph $G$ a sparse TDP-graph with associated partition $S = (S_A, S_B)$.

We now partition the sets $S_A$ and $S_B$ in two different ways depending on the role that the vertices in $S_A$ and $S_B$, respectively, play in the graphs $G_A$ and $G_{AB}$. First, let $S_A = (A_1, A_2, A_3)$ and $S_B = (B_1, B_2, B_3)$ where

- $A_1 = \{v \in S_A \mid d_{G_A}(v) \geq 2\}$
- $A_2 = \{v \in S_A \mid |N_G(v) \cap A_1| \neq 0\}$
- $A_3 = S_A \setminus (A_1 \cup A_2)$

and

- $B_1 = \{v \in S_B \mid d_{G_B}(v) \geq 2\}$
- $B_2 = \{v \in S_B \setminus B_1 \mid |N_G(v) \cap B_1| \neq 0\}$
- $B_3 = S_B \setminus (B_1 \cup B_2)$.

Next, we define a partition of $V(G) = V(G_{AB})$ as the union of the two partitions $S_A = (A_1B, A_2B, A_3B)$ and $S_B = (AB_1, AB_2, AB_3)$ where

- $A_1B = \{v \in S_A \mid d_{G_{AB}}(v) \geq 2\}$
- $A_2B = \{v \in S_A \setminus A_1B \mid v$ has a neighbor in $G_{AB}$ that belongs to $AB_1\}$
- $A_3B = S_A \setminus (A_1B \cup A_2B)$

and

- $AB_1 = \{v \in S_B \mid d_{G_{AB}}(v) \geq 2\}$
- $AB_2 = \{v \in S_B \setminus AB_1 \mid v$ has a neighbor in $G_{AB}$ that belongs to $A_1B\}$
- $AB_3 = S_B \setminus (B_1A \cup B_2A)$.

We note that every vertex in $A_3$ has degree 1 in $G_A$, and every vertex in $A_3B$ has degree 1 in $G_{AB}$. Analogously, every vertex in $B_3$ and $AB_3$ has degree 1 in $G_B$ and $G_{AB}$, respectively. In particular, vertices from $A_3 \cap A_3B$ and from $B_3 \cap AB_3$ have degree 2 in $G$. Further, the neighbor of a vertex from $A_3$ in $G_A$ belongs to $A_3$, and, analogously, the neighbor of a vertex from $B_3$ in $G_B$ belongs to $B_3$. We proceed further with the following series of structural properties of the graph $G$. 

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Claim 2 \( \delta(G) = 2 \).

**Proof.** Recall that \( G \) is a sparse TDP-graph with associated partition \( S = (S_A, S_B) \). Thus, \( S_A \) and \( S_B \) are disjoint total dominating sets of \( G \) which form a partition of \( V(G) \). Every vertex \( v \in V(G) \) has at least one neighbor in \( S_A \) and at least one neighbor in \( S_B \). Hence, \( \delta(G) \geq 2 \). Suppose, to the contrary, that \( \delta(G) > 2 \).

Suppose that \( A_1B \neq \emptyset \) and let \( v \in A_1B \). Let \( v_1, v_2, \ldots, v_k \), where \( k \geq 2 \), be the neighbors of \( v \) in \( G_{AB} \). By Claim 1 and the definition of the set \( AB_2 \), we note that for each \( i \in [k] \), \( v_i \in AB_2 \) and the vertex \( v \) is the only neighbor of \( v_i \) that belongs to the set \( S_A \). Further, since \( d_G(v_i) > 2 \), the vertex \( v \) has at least two neighbors in \( S_B \). By Claim 1, every component of the graph \( G_B \) is a star, implying that no two neighbors of \( v \) are adjacent or have a common neighbor in \( G_B \). Further, every neighbor of \( v_i \) in \( G \) different from \( v \) belongs to the set \( B_2 \), and has the vertex \( v_i \) as its only neighbor in \( G_B \). Thus, the set \( B_2 \) contains at least \( 2k \) vertices at distance 2 from \( v \) in \( G \).

For \( i \in [k] \), let \( w_i \) denote an arbitrary neighbor of \( v_i \) in \( G_B \), and so \( w_i \in B_2 \). Since \( d_G(w_i) > 2 \) and \( w_i \) has only one neighbor in \( S_B \), namely the vertex \( v_i \), we note that \( w_i \in AB_1 \) and therefore \( w_i \) has at least two neighbors in \( A_2B \). By Claim 1 and the definition of the set \( A_2B \), we note that every neighbor of \( w_i \) different from \( v_i \) belongs to the set \( AB_2 \). Further, each such neighbor of \( w_i \) has exactly one neighbor that belongs to the set \( S_B \), namely the vertex \( v_i \), and therefore has at least two vertices in \( S_A \) by the minimum degree condition. By Claim 1, every component of the graph \( G_A \) is a star, and therefore two distinct vertices of degree at least 2 in \( G_A \) belong to different components of \( G_A \). This implies that this subset of \( A_2B \) of vertices in \( S_A \) contains at least \( 4k \) vertices.

By the minimum degree condition, these vertices in \( A_2B \) also belong to \( A_1 \) and each of them has at least two neighbors in \( A_2 \). Further, analogously as before, no two such vertices are the same, implying that this subset of \( A_2 \) contains at least \( 8k - 1 \) vertices distinct from \( v \), all of which belong to the set \( A_1B \), noting that one of these vertices may possibly be the vertex \( v \). By repeating this process for all these vertices we see that we have an infinite process with infinite growth, which is not possible in a finite graph \( G \). Therefore, the set \( A_1B = \emptyset \). Analogously, the set \( AB_1 = \emptyset \).

We now consider a vertex \( v \in A_2B \cup A_3B \). By Claim 1, every component of the graph \( G_{AB} \) is a star, implying that the vertex \( v \) has exactly one neighbor in \( S_B \) and, by the minimum degree condition, at least two neighbors in \( S_A \). Thus, \( v \in A_1 \) and each neighbor of \( v \) in \( S_A \) belong to \( A_2 \). Further, by Claim 1, each such neighbor of \( v \) in \( A_2 \) has degree 1 in \( G_A \) and, therefore, by the minimum degree condition, has at least two neighbors in \( S_B \). Thus, every neighbor of \( v \) in \( A_2 \) belongs to the set \( A_1B \), contradicting our earlier observation that the set \( A_1B \) is an empty set. This completes the proof of Claim 2. \( \square \)

By Claim 2, every sparse TDP-graph has minimum degree 2. In particular, \( \delta(G) = 2 \). Let \( D = \{ v \in V(G) \mid d_G(v) = 2 \} \).

Claim 3 If a vertex in \( D \) is a cut-vertex of \( G \), then \( (G, S) \in \mathcal{G} \).
Proof. Suppose that a vertex in \( D \) is a cut-vertex of \( G \). Suppose firstly that \( D \) contains two adjacent vertices, \( x \) and \( y \), that are both cut-vertices of \( G \), and let \( e = xy \). Let \( C_x \) and \( C_y \) be the components of \( G - e \) which contain \( x \) and \( y \), respectively. Further, let \( x' \) be the neighbor of \( x \) in \( C_x \) and let \( y' \) be the neighbor of \( y \) in \( C_y \). We have two options with respect to the status of the vertices \( x, x', y, y' \). Either \( \text{sta}(x') = \text{sta}(x) \neq \text{sta}(y) = \text{sta}(y') \) or \( \text{sta}(x') = \text{sta}(y') \neq \text{sta}(x) = \text{sta}(y) \). In both cases, let \( G' \) be the graph obtained from \( G - \{x, y\} \) by adding the edge \( x'y' \), and changing the status of all vertices in \( V(C_x) \setminus \{y\} \) while retaining the status of all vertices in \( V(C_y) \setminus \{x\} \). Let \( S' = (S_A', S_B') \) be the resulting partition of \( V(G') \). We note that \( G' \) is a TDP-graph, where \( S' = (S_A', S_B', S') \) is a partition of \( V(G') \) into two total dominating sets. If \( x \) and \( x' \) have the same status in \( G \), then we use Statement 1 with the operation \( O_{18} \) to show that \( (G, S) \in \mathcal{G} \), while if \( x \) and \( x' \) have different status in \( G \), we use Statement 1 with the operation \( O_{19} \).

Thus, we may assume that no two adjacent vertices of \( D \) are both cut-vertices of \( G \). Let \( v \) be a cut-vertex of \( G \) that belongs to \( D \) with neighbors \( u' \) and \( u'' \). Without loss of generality we may assume that \( \text{sta}(u) = \text{sta}(u'') \neq \text{sta}(u') \). Let \( C_{u'} \) and \( C_{u''} \) be the components of \( G - v \) containing \( u' \) and \( u'' \), respectively. Since \( S = (S_A, S_B) \) is a partition of \( V(G) \) into two total dominating sets of \( G \), there exists a neighbor of \( u' \) in \( C_{u'} \) of the same status as \( u' \) and a neighbor of \( u'' \) in \( C_{u''} \) whose status is different from that of \( u'' \). Let \( G' \) be the graph obtained from \( G - v \) by identifying the vertices \( u' \) and \( u'' \) into one new vertex \( u \), and joining \( u \) to every neighbor of \( u' \) and \( u'' \). Further, we assign to \( u \) the same status as that of \( u' \), while we change the status of all vertices in \( V(C_{u''}) \setminus \{u''\} \) and retain the status of all vertices in \( V(C_{u'}) \setminus \{u'\} \). Let \( S' = (S_A', S_B') \) be the resulting partition of \( V(G') \). We note that \( G' \) is a TDP-graph, where \( S' = (S_A', S_B', S) \) is a partition of \( V(G') \) into two total dominating sets. We now use Statement 1 with the operation \( O_{17} \) to show that \( (G, S) \in \mathcal{G} \), where \( H^u_1 = C_{u'} - \{u'\} \) and \( H^u_2 = C_{u''} - \{u''\} \). (2)

By Claim 3, we may assume that no vertex in \( D \) is a cut-vertex of \( G \), for otherwise the desired result follows. We note that every vertex in \( D \) has one neighbor in \( S_A \) and one neighbor in \( S_B \). Further, every component in \( G[D] \) is a path or a cycle.

Claim 4 Let \( C \) be a component of \( G[D] \). If \( C \) is a cycle or if \( C \) is a path of order at least 5 or if \( C \) is a path of order 4 and the ends of \( C \) do not have a common neighbor, then \( (G, S) \in \mathcal{G} \).

Proof. Suppose that \( C \) is a cycle. Since \( G \) is a connected TDP-graph, this implies that \( G \cong C_n \) where \( n \equiv 0 \pmod{4} \). In this case, \( G \) can be obtained from the labeled base graph \( G_1 \) by repeated applications of operation \( O_9 \) (or operation \( O_{10} \)). Hence, we may assume that \( C \) is a path, for otherwise the desired result follows. Let \( C \) be the path \( x_1 x_2 \ldots x_k \), where \( k \geq 4 \). Let \( u \) be the neighbor of \( x_1 \) not on \( C \). If \( k \geq 5 \), let \( v = x_5 \), while if \( k = 4 \), let \( v \) be the neighbor of \( x_4 \) not on \( C \). By assumption, \( u \neq v \). Let \( X = \{x_1, x_2, x_3, x_4\} \).

Suppose firstly that \( \text{sta}(u) = \text{sta}(x_1) \). In this case, \( \text{sta}(x_2) = \text{sta}(x_3) \neq \text{sta}(x_4) = \text{sta}(v) = \text{sta}(x_1) \). If both \( u \) and \( v \) have neighbors of different status in \( G - X \), then let \( G' = G - X \). In this case, the graph \( G' \) is a TDP-graph and we use Statement 1 with the operation \( O_7 \) to show that \( (G, S) \in \mathcal{G} \). Hence, we may assume that the neighbors of \( u \) or \( v \) (or both \( u \)
and v) in G − X all have the same status, which is necessarily different from the status of u. We now let G' be obtained from G − X by adding the edge uv. Once again, the graph G' is a TDP-graph. We use Statement 1 with the operation O_{10} to show that (G, S) ∈ ℜ.

Suppose next that sta(u) ≠ sta(x_1). In this case, sta(x_2) = sta(v) ≠ sta(x_3) = sta(x_4) = sta(u). If both u and v have neighbors of different status in G − X, then let G' = G − X. In this case, the graph G' is a TDP-graph and we use Statement 1 with the operation O_9 to show that (G, S) ∈ ℜ. Hence, we may assume that the neighbors of u or v (or both u and v) in G − X all have the same status. We note that if the neighbors of x, where x ∈ {u, v}, in G − X all have the same status, then this status is the same status as the status of x. We now let G' be obtained from G − X by adding the edge uv. Once again, the graph G' is a TDP-graph. We use Statement 1 with the operation O_9 to show that (G, S) ∈ ℜ. (c)

By Claim 4, we may assume that every component of G[D] is a path-component of order at most 4, and that the ends of a path-component of G[D] of order 4 have a common neighbor in G. In what follows we adopt the following notation. Let P be a path-component of G[D], and so P ≅ P_k where k ∈ [4]. Let P be the path x_1 ... x_k, and let u and v be the vertices in G that do not belong to P and are adjacent to x_1 and x_k, respectively. We call u and v the vertices in G − V(P) associated with the path P. By assumption, if k = 4, then u = v. We note that if k = 1, then u ≠ v. We define next a good path-component.

A good path-component: A path-component P of G[D] is a good path-component if P ≅ P_k where k ∈ [3], and in the graph G− = G − V(P) both u and v have a neighbor that belongs to S_A and a neighbor that belongs to S_B, where u and v are the vertices in G− associated with P.

Claim 5 If G contains a good path-component, then (G, S) ∈ ℜ.

Proof. Suppose that G contains a good path-component, P: x_1 ... x_k. By definition, k ∈ [3]. Suppose that k = 1. Since P is a good path-component, the graph G' = G − x_1 is a TDP-graph. We now use Statement 1 with the operation O_3 to show that (G, S) ∈ ℜ.

Suppose that k = 2. Suppose that sta(x_1) = sta(x_2). Then, sta(u) ≠ sta(x_1) and either u = v or u ≠ v and sta(u) = sta(v). In both cases, since P is a good path-component, the graph G' = G − V(P) is a TDP-graph. If u = v, we use Statement 1 with the operation O_{11} to show that (G, S) ∈ ℜ, while if u ≠ v, we use Statement 1 with the operation O_4 to show that (G, S) ∈ ℜ. Suppose that sta(x_1) ≠ sta(x_2). Then, sta(u) = sta(x_1) and sta(v) = sta(x_2). Since P is a good path-component, the graph G' = G − V(P) is a TDP-graph, and we use Statement 1 with the operation O_5 to show that (G, S) ∈ ℜ.

Suppose that k = 3. Without loss of generality we may assume that sta(x_1) ≠ sta(x_2) = sta(x_3), implying that sta(u) = sta(x_1) and either u = v or u ≠ v and sta(u) = sta(v). Since P is a good path-component, the graph G' = G − V(P) is a TDP-graph. If u = v, we use Statement 1 with the operation O_{12} to show that (G, S) ∈ ℜ, while if u ≠ v, we use Statement 1 with the operation O_6 to show that (G, S) ∈ ℜ. (c)
By Claim 5, we may assume that $G$ contains no good path-component, for otherwise the desired result follows. Hence, if $P$ is a component of $G[\bar{D}]$, then $P \cong P_4$ and the ends of $P$ have a common neighbor in $G$ or $P$ is not a good path-component. We define next a good cut-vertex and a good bridge.

**A good cut-vertex:** Let $u$ be a cut-vertex in $G$, and let $H_1$ and $H_2$ be the associated subgraphs of $G - u$, where $H_2$ is chosen to be a component of $G - u$ of minimum order. Since $\delta(G) = 2$ by Claim 2, we note that $H_2^u$ contains at least three vertices. The cut-vertex $u$ is a good cut-vertex if $H_2^u$ contains at most five vertices and in the graph $H_1^u$, the vertex $u'$ has a neighbor that belongs to $S_A$ and a neighbor that belongs to $S_B$.

**A good bridge:** Let $e = vx$ be a bridge in $G$, and let $G_v$ and $G_x$ be the two components of $G - e$. We note that $G_v = G[W_{vx}]$ and $G_x = G[W_{xv}]$. The bridge $e = vx$ is a good bridge if one component of $G - e$, say $G_x$, contains at most five vertices and in the other component $G_v$, the vertex $v$ has a neighbor that belongs to $S_A$ and a neighbor that belongs to $S_B$.

We show next that if $G$ has a good cut-vertex or a good bridge, then $(G, S) \in \mathcal{G}$, as desired.

**Claim 6** If $G$ contains a good cut-vertex or a good bridge, then $(G, S) \in \mathcal{G}$.

**Proof.** Recall that if $u$ is a good cut-vertex of $G$, then $3 \leq n(H_2^u) \leq 5$, and if $e = vx$ is a good bridge in $G$, then $3 \leq n(G_x) \leq 5$. We proceed further with a series of six subclaims.

**Claim 6.1** If $G$ contains a good cut-vertex $u$ with $n(H_2^u) = 3$, then $(G, S) \in \mathcal{G}$.

**Proof.** Suppose that $u$ is a good cut-vertex in $G$ with $n(H_2^u) = 3$. Since $\delta(G) = 2$, we note that $H_2^u \cong C_3$. Let $V(H_2^u) = \{u', x, y\}$. Since $G$ is a TDP-graph, we note that $\text{sta}(x) = \text{sta}(y) \neq \text{sta}(u)$. Since $u$ is a good cut-vertex, the vertex $u'$ has a neighbor in $S_A$ and a neighbor in $S_B$ in $H_1^u$, implying that the graph $G' = G - \{x, y\}$ is a TDP-graph. We now use Statement 1 with the operation $O_{11}$ to show that $(G, S) \in \mathcal{G}$. (c)

**Claim 6.2** If $G$ contains a good bridge $e = vx$ with $n(G_x) = 3$, then $(G, S) \in \mathcal{G}$.

**Proof.** Suppose that $e = vx$ is a good bridge of $G$ with $n(G_x) = 3$. Since $\delta(G) = 2$, we note that $G_x \cong C_3$. Let $V(G_x) = \{x, y, z\}$. Since $G$ is a TDP-graph, we note that $\text{sta}(x) = \text{sta}(v) \neq \text{sta}(y) = \text{sta}(z)$. Since $e$ is a good bridge, the vertex $v$ has a neighbor in $S_A$ and in $S_B$ in $G_v$. Therefore, $G' = G - \{x, y, z\}$ is a TDP-graph. We now use Statement 1 with the operation $O_{14}$ to show that $(G, S) \in \mathcal{G}$. (c)
Claim 6.3 If $G$ contains a good cut-vertex $u$ with $n(H_2^u) = 4$, then $(G, S) \in \mathcal{G}$.

Proof. Suppose that $u$ is a good cut-vertex in $G$ with $n(H_2^u) = 4$. Let $V(H_2^u) = \{u'', x, y, z\}$. If $u''$ has only one neighbor in $H_2^u$, say $x$, then $G[\{x, y, z\}] \cong C_3$ since $\delta(G) = 2$. This implies that since $u$ is a good cut-vertex, the edge $ux$ is a good bridge with $n(G_x) = 3$, and so, by Claim 6.2, $(G, S) \in \mathcal{G}$. Hence, we may assume that $u''$ has at least two neighbors in $H_2^u$.

Suppose that $u''$ is adjacent to all three vertices $x, y$ and $z$. In this case, $H_2^u$ is isomorphic to $K_4$ or to $K_4 - e$. If $H_2^u \cong K_4 - e$, we may assume, remaining vertices if necessary, that $e = yz$. In both cases, if $H_2^u$ has two vertices of the same status and the remaining two of a different status, then since the vertices of $H_2^u$ induce a path $P_4$ in the graph $G_{AB}$, we contradict Claim 1. Hence, since $G$ is a TDP-graph, this implies that sta($u''$) $\neq$ sta($x$) $= sta(y) = sta(z)$. Since $G$ is a sparse TDP-graph, this implies that $H_2^u \cong K_4 - e$, and therefore that $\{y, z\} \subseteq D$. However, then the path $P$ with $V(P) = \{y\}$ is a good path-component, contradicting our assumption that $G$ contains no good path-component.

Therefore, $u''$ is adjacent in $H_2^u$ to exactly two vertices, say $x$ and $z$. Since $\delta(G) = 2$, the vertex $y$ has degree 2 and hence belongs to $D$. Further, $y$ is adjacent to both $x$ and $z$. Thus, sta($x$) $\neq$ sta($z$), which in turn implies that sta($u''$) $\neq$ sta($y$) since $G$ is a TDP-graph. If $x$ and $z$ are adjacent, then the vertices of $H_2^u$ induce a path $P_4$ in the graph $G_{AB}$, a contradiction. Therefore, $x$ and $z$ are not adjacent, and so $H_2^u$ is the 4-cycle $u''xyzu''$. Without loss of generality we may assume that sta($x$) $=$ sta($u''$) $\neq$ sta($y$) $=$ sta($z$). Since $u$ is a good cut-vertex, the vertex $u'$ has a neighbor in $S_A$ and in $S_B$ in $H_2^1$. Therefore, $G' = G - \{x, y, z\}$ is a TDP-graph. We now use Statement 1 with the operation $\mathcal{O}_{12}$ to show that $(G, S) \in \mathcal{G}$. (c)

Claim 6.4 If $G$ contains a good bridge $e = vx$ with $n(G_x) = 4$, then $(G, S) \in \mathcal{G}$.

Proof. Suppose that $e = vx$ is a good bridge of $G$ with $n(G_x) = 4$. Let $V(G_x) = \{x, y, z, w\}$. Analogously as in the proof of Claim 6.3, noting that $G$ is a sparse graph and $\delta(G) = 2$, we may assume that $G_x \cong C_4$, for otherwise we get a contradiction or deduce that $(G, S) \in \mathcal{G}$. Renaming vertices, if necessary, we may assume that $G_x$ is the 4-cycle $xyzwx$. Without loss of generality we may assume that sta($x$) $=$ sta($y$) $\neq$ sta($z$) $=$ sta($w$). Since $e = vx$ is a good bridge, the vertex $v$ has a neighbor in $S_A$ and in $S_B$ in $G_v$. Therefore, $G' = G - \{x, y, z\}$ is a TDP-graph. We now use Statement 1 with the operation $\mathcal{O}_{15}$ to show that $(G, S) \in \mathcal{G}$. (c)

Claim 6.5 If $G$ contains a good cut-vertex, then $(G, S) \in \mathcal{G}$.

Proof. Suppose that $G$ contains a good cut-vertex $u$. By Claims 6.1 and 6.3, we may assume that $n(H_2^u) = 5$, for otherwise $(G, S) \in \mathcal{G}$, as desired. Let $V(H_2^u) = \{u', x, y, z, w\}$. Let $U = \{x, y, z, w\}$ and let $G_U = G[U]$. By our choice of $H_2$ as a component of $G - u$ of minimum order, the graph $G_U$ is connected. Further, since $\delta(G) = 2$, the vertex $u''$ is adjacent in $H_2^u$ to every vertex of $U$ of degree 1 in $G_U$. 

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Claim 6.5.1 If $G_U \cong P_4$, then $(G, S) \in \mathcal{G}$.

Proof. Suppose that $G_U \cong P_4$. Renaming vertices if necessary, we may assume that $G_U$ is the path $xyzw$. As observed earlier, $u''$ is therefore adjacent in $H^u_2$ to both vertices of degree 1 in $G_U$, namely to $x$ and $w$. Suppose that $u''$ is adjacent to at least one of $y$ or $z$ in $H^u_2$. By symmetry, we may assume that $u''$ is adjacent to $z$. Since both $x$ and $w$ have a neighbor in $S_A$ and in $S_B$, we note that $\text{sta}(y) = \text{sta}(z) \neq \text{sta}(u'')$. Since $u$ is a good cut-vertex, the vertex $u''$ has a neighbor in $S_A$ and in $S_B$ in $H^u_2$. These observations imply that the path $P$ with $V(P) = \{w\}$ is a good path-component, contradicting our assumption that $G$ contains no good path-component. Hence, $u''$ is adjacent only to $x$ and $w$ in $H^u_2$. Since $G$ is a TDP-graph, we note that $\text{sta}(u'') = \text{sta}(x) = \text{sta}(w) \neq \text{sta}(y) = \text{sta}(z)$. Since $u$ is a good cut-vertex, the graph $G' = G - \{x, y, z, w\}$ is therefore a TDP-graph. We now use Statement 1 with the operation $\mathcal{O}_{13}$ to show that $(G, S) \in \mathcal{G}$. (c)

Claim 6.5.2 $G_U \not\cong K_{1,3}$.

Proof. Suppose, to the contrary, that $G_U \cong K_{1,3}$. Renaming vertices if necessary, we may assume that $w$ is the center of the star $G_U$, implying that $u''$ is adjacent in $H^u_2$ to $x$, $y$, and $z$. The edge $u''w$ may or may not be present. Since every vertex of degree 2 has a neighbor in $S_A$ and in $S_B$, we note that $\text{sta}(u'') \neq \text{sta}(w)$. At least two vertices in $\{x, y, z\}$ have the same status. Renaming vertices, if necessary, that $\text{sta}(x) = \text{sta}(y)$. Since $w$ has a neighbor in $S_A$ and in $S_B$ in $G$, so too does $w$ have a neighbor in $S_A$ and in $S_B$ in $G - x$. These observations, together with the fact that $u$ is a good cut-vertex, imply that the path $P$ with $V(P) = \{x\}$ is a good path-component, contradicting our assumption that $G$ contains no good path-component. Therefore, $G_U \not\cong K_{1,3}$. (c)

Claim 6.5.3 If $G_U \cong C_4$, then $(G, S) \in \mathcal{G}$.

Proof. Suppose that $G_U \cong C_4$. Renaming vertices if necessary, we may assume that $G_U$ is the cycle $xyzux$. We show that $u''$ is adjacent only to one vertex in $H^u_2$. Suppose, to the contrary, that $u''$ has degree at least 2 in $H^u_2$, for otherwise the desired result follows. Suppose that $\text{sta}(w) = X$.

Suppose that $u''$ is adjacent to two adjacent vertices of $U$, say $x$ and $y$. Suppose that $\text{sta}(z) = \overline{X}$. If in this case, $\text{sta}(x) = \overline{X}$, then, since $w$ has a neighbor in $S_A$ and in $S_B$, the vertex $u''$ is adjacent to $u''$ and $\text{sta}(u'') = X$. This implies that $\text{sta}(y) = \overline{X}$ since $x$ has a neighbor in $S_A$ and in $S_B$. But then $wxu''y$ is a path $P_4$ in $G_{AB}$, contradicting Claim 1. Hence, $\text{sta}(x) = \text{sta}(w) = X$. Analogously, $\text{sta}(y) = \text{sta}(z) = \overline{X}$. Recall that the vertex $u$ is a good cut-vertex, and so has neighbors $v_1$ and $v_2$ of status $X$ and $\overline{X}$, respectively, that belong to $V(H_1)$. If $\text{sta}(u) = X$, then $wxuv_1$ is a path $P_4$ in the graph $G_X$, while if $\text{sta}(u) = \overline{X}$, then $zyuv_2$ is a path $P_4$ in the graph $G_{\overline{X}}$. Both cases contradict Claim 1. Hence, $\text{sta}(z) = X$.

Suppose now that $\text{sta}(x) = X$. If $\text{sta}(u'') = X$ or $\text{sta}(y) = X$, then $G_X$ contains a path $P_4$, a contradiction. Hence, $\text{sta}(u'') = \text{sta}(y) = \overline{X}$. But then $zyzu$ is a path $P_4$ in the graph
G_{AB}$, a contradiction. Hence, $\text{sta}(x) = \overline{X}$. Analogously, $\text{sta}(y) = \overline{X}$. If $\text{sta}(u'') = \overline{X}$, then $xu''yx$ is a 3-cycle in $G_{\overline{X}}$, while if $\text{sta}(u'') = X$, then $wzu''y$ is a path $P_4$ in $G_{AB}$. Both cases contradict Claim 1. Therefore, no two consecutive vertices on the 4-cycle $G_U$ are both adjacent vertices to $u''$, implying that $H_2^n \cong K_{2,3}$.

Renaming vertices of $U$, if necessary, we may assume that the two neighbors of $u''$ that belong to $U$ are $x$ and $z$. Thus, $\{y, w\} \subseteq D$. Renaming $x$ and $z$, if necessary, we may assume that $\text{sta}(x) = \overline{X}$ and $\text{sta}(z) = X$. If $\text{sta}(y) = X$, then $\text{sta}(u) = \overline{X}$, implying that the path $P$ with $V(P) = \{y\}$ is a good path-component. If $\text{sta}(y) = \overline{X}$ and $\text{sta}(u) = X$, then the path $P$ with $V(P) = \{y\}$ is a good path-component. If $\text{sta}(y) = \overline{X}$ and $\text{sta}(u) = X$, then the path $P$ with $V(P) = \{w\}$ is a good path-component. In all cases, we contradict our assumption that $G$ contains no good path-component.

Therefore, $u''$ is adjacent only to one vertex in $H_2^u$, say $x$. In this case, $vx$ is a good bridge with $n(G_x) = 4$, and so, by Claim 6.4, $(G, S) \in \mathcal{G}$. ($\Box$)

**Claim 6.5.4** If $G_U$ is a $K_3$ with a pendant edge, then $(G, S) \in \mathcal{G}$.

**Proof.** Suppose that $G_U$ is $K_3$ with a pendant edge. Renaming vertices if necessary, we may assume that $yzwy$ is the cycle and $xy$ the pendant edge in $G_U$, implying that $u''$ is adjacent to $x$ in $H_2^u$. If $x$ is the only neighbor of $u''$ in $H_2^u$, then $x$ is a cut-vertex of degree 2 in $G$, contradicting our assumption that no vertex in $D$ is a cut-vertex of $G$. Hence, $u''$ is adjacent to at least one of $w$, $y$, and $z$ in $H_2^u$. Let $\text{sta}(y) = X$, and so $\text{sta}(u'') = \overline{X}$ since $x$ has a neighbor in $S_A$ and in $S_B$.

Suppose that $u''$ is in addition adjacent only to $y$ in $H_2^u$. Then, $\text{sta}(w) = \text{sta}(z) = \overline{X}$ and $\text{sta}(x) = X$, since each vertex in $U$ has a neighbor in $S_A$ and in $S_B$. This implies that $y$ is a good cut-vertex with $|V(H_2^u)| = 3$, and so, by Claim 6.1, $(G, S) \in \mathcal{G}$. Hence, we may assume that $u''$ is adjacent to at least one of $z$ and $w$, say $w$, for otherwise the desired result follows.

Suppose that $u''$ is adjacent only to $x$ and $w$ in $H_2^u$. In this case, $\text{sta}(w) = \overline{X}$ since $z$ has a neighbor in $S_A$ and in $S_B$. If $\text{sta}(x) = \overline{X}$, then $\text{sta}(z) = X$, and so $zwxy$ is a path $P_4$ in $G_{AB}$, contradicting Claim 1. If $\text{sta}(x) = X$, then the path $P$ with $V(P) = \{z\}$ is a good path-component, contradicting our assumption that $G$ contains no good path-component. Hence, $u''$ is adjacent to at least one of $y$ and $z$ in $H_2^u$.

Suppose that $u''$ is adjacent to $z$ but not to $y$ in $H_2^u$. If $\text{sta}(w) = \text{sta}(z)$, then either $wyzw$ is a 3-cycle in $G_X$ or $wzu''w$ is a 3-cycle in $G_{\overline{X}}$ depending on whether $\text{sta}(w) = X$ or $\text{sta}(w) = \overline{X}$, respectively. Both cases contradict Claim 1. Therefore, $\text{sta}(w) \neq \text{sta}(z)$. By symmetry, we may assume that $\text{sta}(w) = \overline{X}$ and $\text{sta}(z) = X$. But then $gwyzu''$ is a path $P_4$ in $G_{AB}$, contradicting Claim 1.

Suppose that $u''$ is adjacent to $y$ but not to $z$ in $H_2^u$. In this case, $\text{sta}(w) = \overline{X}$ since $z$ has a neighbor in $S_A$ and in $S_B$. If $\text{sta}(x) = \overline{X}$, then $\text{sta}(z) = X$, and so $zwxy$ is a path $P_4$ in $G_{AB}$. If $\text{sta}(x) = X$, then the path $P$ with $V(P) = \{z\}$ is a good path-component. Both cases produce a contradiction.
Therefore, $u''$ is adjacent to both $y$ and $z$ in $H_2^v$. If $\text{sta}(w) = \text{sta}(z)$, then either $wyzw$ is a 3-cycle in $G_X$ or $wzu''w$ is a 3-cycle in $G_X^\sim$, a contradiction. Therefore, $\text{sta}(w) \neq \text{sta}(z)$. By symmetry, we may assume that $\text{sta}(w) = \overline{X}$ and $\text{sta}(z) = X$. But then $ywu''u$ is a path $P_4$ in $G_{AB}$, contradicting Claim 1. (c)

**Claim 6.5.5** $G_U \not\cong K_4 - e$.

**Proof.** Suppose, to the contrary, that $G_U \cong K_4 - e$. Renaming vertices if necessary, we may assume that $e = yw$. Let $\text{sta}(x) = X$. Suppose that $\text{sta}(z) = X$. If $\text{sta}(y) = X$, then $xyzx$ is a 3-cycle in $G_X$, a contradiction. Hence, $\text{sta}(y) = \overline{X}$. Analogously, $\text{sta}(w) = \overline{X}$. But then $xyzw$ is a path $P_4$ in $G_{AB}$, a contradiction. Hence, $\text{sta}(z) = \overline{X}$. Suppose that $\text{sta}(y) \neq \text{sta}(w)$. By symmetry, we may assume that $\text{sta}(y) = X$ and $\text{sta}(w) = \overline{X}$. But then $yzzw$ is a path $P_4$ in $G_{AB}$, a contradiction. Hence, by symmetry, we may assume that $\text{sta}(y) = \text{sta}(w) = X$, implying that $z$ is adjacent to $u''$ in $H_2^v$ and $\text{sta}(u'') = \overline{X}$. If $u''$ is adjacent to any other vertex of $H_2^v$, then there exists a path $P_4$ in $G_{AB}$, a contradiction. Hence, $u''$ is adjacent only to $z$ in $H_2^v$, implying that the path $P$ with $V(P) = \{y\}$ is a good path-component, a contradiction. Therefore, $G_U \not\cong K_4 - e$. (c)

**Claim 6.5.6** $G_U \not\cong K_4$.

**Proof.** Suppose, to the contrary, that $G_U \cong K_4$. If three vertices in $U$ have the same status, then these three vertices induce a 3-cycle in $G_A$ or $G_B$, a contradiction. Hence, two vertices in $U$ have the same status and the remaining two vertices have a different status. But this implies that there exists a path $P_4$ in $G_{AB}$, a contradiction. Therefore, $G_U \not\cong K_4$. (c)

By Claim 6.5.1 to Claim 6.5.6, the graph $(G, S) \in \mathcal{G}$. This completes the proof of Claim 6.5. (c)

**Claim 6.6** *If $G$ contains a good bridge, then $(G, S) \in \mathcal{G}$.***

**Proof.** Suppose that $e = vx$ is a good bridge of $G$. By Claims 6.2 and 6.4, we may assume that $n(G_x) = 5$, for otherwise $(G, S) \in \mathcal{G}$, as desired. Let $V(G_x) = \{x, y, z, w, t\}$. If $d_G(x) = 2$, then $x$ is a cut-vertex of $G$, contradicting our earlier assumption that no vertex in $D$ is a cut-vertex of $G$. Thus, $d_G(x) \geq 3$. Renaming vertices if necessary, we may assume that $y$ and $t$ are neighbors of $x$.

If $x$ is a cut-vertex of $G_x$, then $G_X$ consists of two triangles that have one vertex in common, namely the vertex $x$. In this case, letting $\text{sta}(x) = X$, we note that all four other vertices in $G_x$ have status $\overline{X}$, implying that $\text{sta}(u) = X$ and therefore that $x$ is a good cut-vertex of $G$ with $|H_2^x| = 3$. Thus, by Claim 6.1, $(G, S) \in \mathcal{G}$. Hence, we may assume that $x$ is not a cut-vertex of $G_x$. Thus, letting $U = \{y, z, w, t\}$ and $G_U = G[U]$, the graph $G_U$ is connected. We now proceed exactly as in the proof of Claim 6.5 to show that the graph $(G, S) \in \mathcal{G}$, except in one exceptional case where additional work is needed. This
exceptional case occurs in the second paragraph of the proof of Claim 6.5.3 where we rely on the fact that \( u \) is a good cut-vertex, and hence deduce the existence of a path \( P_4 \) in \( G_X \) or in \( G_X' \).

In order to handle this exceptional case, we may assume that \( G_U \cong C_4 \), where \( yzwty \) is the 4-cycle in \( G_U \), and that \( x \) is adjacent to two adjacent vertices of \( U \), say \( y \) and \( z \). Further, we may assume that \( \text{sta}(y) = \text{sta}(t) = X \) and \( \text{sta}(z) = \text{sta}(w) = X \). By symmetry, we may assume that \( \text{sta}(x) = X \). If \( \text{sta}(v) = X \), then \( vxyt \) is a path \( P_4 \) in \( G_X \). If \( \text{sta}(v) = \overline{X} \), then \( vxzy \) is a path \( P_4 \) in \( G_{AB} \). Both cases produce a contradiction. Hence, this exceptional case cannot occur, implying that analogously as in the proof of Claim 6.5, the graph \((G, S) \in \mathcal{G}\).

This completes the proof of Claim 6.6. (c)

Claim 6.1 to Claim 6.6 exhaust all the cases when \( G \) contains a good cut-vertex or a good bridge. This completes the proof of Claim 6. (c)

By Claim 6, we may assume that \( G \) does not contain a good cut-vertex or a good bridge, for otherwise \((G, S) \in \mathcal{G}\), and the desired result follows. We define next a mixed path-component.

A mixed path-component: A path-component \( P \) of \( G[D] \) is a mixed path-component if \( P \cong P_k \) where \( k \in [3] \), and in the graph \( G^- = G - V(P) \) exactly one of \( u \) and \( v \) has all its neighbors of the same status, where \( u \) and \( v \) are the vertices in \( G^- \) associated with \( P \). Further, we call the vertex in \( G^- \) associated with \( P \) that has all its neighbors in \( G^- \) of the same status a bad vertex and we call the other vertex in \( G^- \) associated with \( P \) that has a neighbor in \( S_A \) and a neighbor in \( S_B \) in \( G^- \) a good vertex. Thus, if \( P \) is a mixed path-component, then one of the vertices in \( G^- \) associated with \( P \) is a bad vertex and the other is a good vertex.

Recall that by our early assumptions, the graph \( G \) is connected and \( \delta(G) \geq 2 \). Further, every component of \( G[D] \) is a path-component of order at most 4, and that the ends of a path-component of \( G[D] \) of order 4 have a common neighbor in \( G \). In particular, \( G \) is not a cycle, and so \( \Delta(G) > 2 \). Recall that by our earlier assumptions, \( G \) contains no good path-component.

Claim 7 If \( G \) is a 2-connected graph, then every path-component of \( G[D] \) is a mixed path-component.

Proof. Let \( G \) be a 2-connected graph. Thus, the ends of a path-component of \( G[D] \) do not have a common neighbor in \( G \), implying that every component \( P \) of \( G[D] \) is a path-component of order at most 3 and the vertices in \( G - V(P) \) associated with \( P \) are distinct. Let \( Q_1, \ldots, Q_\ell \) be the path-components of \( G[D] \). Let \( u_i \) and \( v_i \) be the two vertices of \( G - V(Q_i) \) associated with the path \( Q_i \) for \( i \in [\ell] \), and let \( Q_i \) be an \((a_i, b_i)\)-path where \( a_i \) is adjacent to \( u_i \) and \( b_i \) is adjacent to \( v_i \) in \( G \). Since \( G \) contains no good path-component, at least one \( u_i \) and \( v_i \) is a bad vertex for each \( i \in [\ell] \).
Suppose, to the contrary, that some path-component of \( G[D] \) is not a mixed path-component. Renaming components if necessary, we may assume that \( Q_1 \) is not a mixed path-component. Since there is no good path-component, both \( u_1 \) and \( v_1 \) are bad vertices.

If \( \ell = 1 \), let \( a = a_1 \) and let \( b = b_1 \), and let define \( v_2 = v_1 \). (If \( Q_1 \) has order 1, then we note that \( a = b \).) If \( \ell \geq 2 \), then let \( a = a_1 \) and let \( b = b_2 \). Since \( G \) is 2-connected, there exists two internally disjoint \((a,b)\)-paths, say \( Q \) and \( R \). Renaming the paths \( Q \) and \( R \), if necessary, interchanging the names of the vertices \( a_i \) and \( b_i \), if necessary, for \( i \in [\ell] \setminus \{1\} \), we may assume that if \( \ell \geq 2 \), the path \( Q_2 \) and the vertices \( a_2 \) and \( b_2 \) are chosen so that \( Q \) is a shortest \((a,b)\)-path that contains \( u_1 \) and \( v_2 \) with the property that all inner vertices of \( Q \) have degree at least 3 in \( G \). Thus, the path \( R \) contains both paths \( Q_1 \) and \( Q_2 \) as subpaths, as well as both vertices \( v_1 \) and \( u_2 \). Let \( Q \) be the path \( q_0 q_1 \ldots q_k q_{k+1} \), where \( q_0 = a, q_1 = u_1, q_k = v_2 \) and \( q_{k+1} = b \), and let \( c_j \) be a neighbor of \( q_j \) not on \( Q \) for \( j \in [k] \). Let \( R' \) be the \((b_1, a_2)\)-subpath of \( R \).

**Claim 7.1** The vertex \( v_2 \) is a good vertex. Further, \( \ell \geq 2 \).

**Proof.** Let \( \text{sta}(q_0) = X \). Since \( Q_1 \) is neither a good nor a mixed path-component, \( q_1 \) is a bad vertex and all its neighbors in \( G - V(Q_1) \) therefore have the same status which is necessarily status \( X \) since \( G \) is a TDP-graph. In particular, \( \text{sta}(c_1) = \text{sta}(q_2) = X \).

Suppose first that \( \text{sta}(q_1) = X \). If \( q_2 \) has a neighbor, \( z_2 \) say, different from \( q_1 \), of status \( X \), then \( c_1 q_1 q_2 z_2 \) is a path \( P_4 \) in \( G_{AB} \), a contradiction. Hence, all neighbors of \( q_2 \) different than \( q_1 \) must have the same status as \( q_2 \), namely \( \overline{X} \). In particular, \( \text{sta}(c_2) = \text{sta}(q_3) = \overline{X} \). If \( q_3 \) has a neighbor, \( z_3 \) say, different than \( q_2 \), of status \( \overline{X} \), then either \( z_3 = c_2 \), in which case \( c_2 q_2 q_3 \overline{c_2} \) is a 3-cycle in \( G_{\overline{X}} \), or \( z_3 \neq c_2 \), in which case \( c_2 q_2 q_3 z_3 \) is a path \( P_4 \) in \( G_{\overline{X}} \). Both cases produce a contradiction. Hence, all neighbors of \( q_3 \) different than \( q_2 \) have status \( X \). In particular, \( \text{sta}(c_3) = \text{sta}(q_4) = X \). Continuing this way, we see that the status of all vertices of the path \( Q \) and their neighbors is forced. In particular, for \( i \geq 0 \), we note that \( q_{4i+1} \) and \( q_{4i+3} \) have status \( X \) and \( q_{4i+1} \) and \( q_{4i+3} \) have status \( \overline{X} \). Further, \( \text{sta}(q_{i-1}) \neq \text{sta}(c_i) \) for \( i \in [k] \). Since \( \text{sta}(q_{k-1}) \neq \text{sta}(c_k) \), the vertex \( q_k \) is a good vertex. If \( \ell = 1 \), then \( q_1 = v_1 \) and the path \( Q_1 \) is a mixed path-component, a contradiction. Hence, \( \ell \geq 2 \), implying that \( v_2 = q_k \) is a good vertex.

Suppose next that \( \text{sta}(q_1) = \overline{X} \). If \( q_2 \) has a neighbor, \( z_2 \) say, different from \( q_1 \), of status \( \overline{X} \), then either \( z_2 = c_1 \), in which case \( c_1 q_1 q_2 c_1 \) is a 3-cycle in \( G_{\overline{X}} \), or \( z_2 \neq c_1 \), in which case \( c_1 q_1 q_2 z_2 \) is a path \( P_4 \) in \( G_{\overline{X}} \). Both cases produce a contradiction. Hence, all neighbors of \( q_2 \) different than \( q_1 \) have status \( \overline{X} \). In particular, \( \text{sta}(c_2) = \text{sta}(q_3) = \overline{X} \). If \( q_3 \) has a neighbor, \( z_3 \) say, different than \( q_2 \), of status \( \overline{X} \), then \( c_2 q_2 q_3 z_3 \) is a path \( P_4 \) in \( G_{\overline{X}} \), a contradiction. Hence, all neighbors of \( q_3 \) different than \( q_2 \) have status \( \overline{X} \). In particular, \( \text{sta}(c_3) = \text{sta}(q_4) = \overline{X} \). Continuing this way, we see that the status of all vertices of the path \( Q \) and their neighbors is forced. In particular, for \( i \geq 0 \), we note that \( q_{4i+1} \) and \( q_{4i+3} \) have status \( \overline{X} \) and \( q_{4i+1} \) and \( q_{4i+3} \) have status \( X \). Further, \( \text{sta}(q_{i-1}) \neq \text{sta}(c_i) \) for \( i \in [k] \). Since \( \text{sta}(q_{k-1}) \neq \text{sta}(c_k) \), the vertex \( q_k \) is a good vertex. Analogously as in the previous case when \( \text{sta}(q_1) = X \), this implies that \( \ell \geq 2 \) and that \( v_2 = q_k \) is a good vertex. (c)

**Claim 7.2** The vertex \( u_2 \) is a good vertex.
Proof. We consider the path $R'$. Let $R'$ be the path $r_0 r_1 \ldots r_p r_{p+1}$, where $r_0 = b_1$, $r_1 = v_1$, $r_p = u_2$ and $r_{p+1} = a_2$. Further, if $d_G(r_j) \geq 3$, let $c_j$ be a neighbor of $r_j$ not on $R'$ for $j \in [p]$. If all internal vertices on $R'$ have degree at least 3, then, analogously as in the proof of Claim 7.1, the vertex $u_2$ is a good vertex. Hence, we may assume that at least one internal vertex of $R'$ has degree 2. Let $r_{t+1}$ be the first internal vertex on $R'$ with $d_G(r_{t+1}) = 2$. Since $r_1$ and $r_p$, both have degree at least 3, we note that $1 < t < p$. Analogous arguments as in the proof of Claim 7.1 show that the vertex $r_t$ is a good vertex. Renaming components $Q_3, \ldots, Q_t$, if necessary, we may assume that $r_{t+1}$ belongs to the path-component $Q_3$, and that $r_{t+1} = a_3$. Since $Q_3$ is not a good path-component, the vertex $v_3$, which immediately following the vertex $b_3$ on $R'$, is a bad vertex. We now consider the $(v_3, u_2)$-subpath of $R'$. Continuing in this manner, we eventually show that the vertex $u_2 = r_p$ is a good vertex. (\(\ast\))

By Claim 7.1 and Claim 7.2, both vertices $u_2$ and $v_2$ are good vertices, implying that the path-component $Q_2$ is good. This contradicts our assumption that there is no good path-components, and completes the proof of Claim 7. (\(\ast\))

As an immediate consequence of the proof of Claim 7, we have the following claim.

**Claim 8** Let $G$ be a 2-connected graph, and let $u_i$ and $v_j$ be distinct vertices associated with path-components in $G[D]$. If all inner vertices of a $(u_i, v_j)$-path, $P$, have degree at least 3 in $G$, then one of $u_i$ and $v_j$ is a good vertex and the other a bad vertex. Moreover, if $u_i$ is a bad vertex, then the status of the inner vertices of $P$ depend only on the status of $u_i$ and on the status of its neighbor in $D$.

We are now in a position to prove the following structural property of path-components.

**Claim 9** If $G$ is a 2-connected graph, then all path-component of $G[D]$ lie on a common cycle.

**Proof.** Let $G$ be a 2-connected graph. Among all cycles in $G$, let $C$ be chosen to contain the maximum number of path-component of $G[D]$. Following the notation introduced in the proof of Claim 7, let $Q_1, \ldots, Q_t$ be the path-components of $G[D]$. Further, let $Q_i$ be a $(a_i, b_i)$-path and let $u_i$ and $v_i$ be the two vertices of $G - V(Q_i)$ associated with the path $Q_i$ for $i \in [\ell]$, where $u_i$ is adjacent to $a_i$ and $v_i$ is adjacent to $b_i$.

Suppose that $C$ contains $t$ path-components of $G[D]$. If $t = \ell$, then $C$ contains all path-components of $G[D]$ and the result is immediate. Hence, we may assume that $t < \ell$. Renaming components and vertices, if necessary, we may assume that the cycle $C$ contains $Q_1, \ldots, Q_t$, where $Q_i+1$ follows $Q_i$ on $C$ for $i \in [t-1]$. Further, renaming $u_i$ and $v_i$, if necessary, we may assume that $C$ starts at $u_1$, proceeds along the edge $u_1 a_1$, follows $Q_1$ from $a_1$ to $b_1$, and then proceeds along the edge $b_1 v_1$ to $v_1$. Thereafter, we assume that $u_{i+1}$ follows $v_i$ for every $i \in [t-1]$ and every internal vertex of this $(v_i, u_{i+1})$-subpath contains no vertex of degree 2. By Claim 7, every component of $G[D]$ is a mixed path-component. Thus, from our ordering of the cycle $C$, either all $u_i$’s are good vertices and all $v_i$’s are bad vertices or vice versa for every $i \in [\ell]$. 

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We now consider the path-component \( Q_{i+1} \). Let \( P' \) be a shortest path in \( G \) from \( u_{i+1} \) to a vertex of \( C \), say \( w \), and let \( P'' \) be a shortest path in \( G \) from \( v_{i+1} \) to a vertex of \( C \), say \( z \). Let \( P \) be the \((w, z)\)-path obtained from following \( P' \) in the reverse direction from \( w \) to \( u_{i+1} \), proceeding along the path \( Q_{i+1} \), and then following the path \( P'' \) from \( v_{i+1} \) to \( z \). Thus, \( w \) and \( z \) are the only vertices on \( P \) that belong to the cycle \( C \). Let \( Q_i \) be the component of \( G[D] \) that proceeds \( w \) on \( C \) and let \( Q_j \) be the component of \( G[D] \) that precedes \( z \) on \( C \).

Suppose first that \( v_i \) is a good vertex. Thus, all \( u_s \)'s are bad vertices and all \( v_s \)'s are good vertices for \( s \in [t] \). In particular, \( v_j \) is a good vertex. However, applying Claim 7 and Claim 8, traversing \( C \) from \( v_i \) to \( w \), then continuing along the path \( P \) to \( z \), and then following the path from \( z \) to \( v_j \) in \( C - v_i \), we deduce that \( v_j \) is a bad vertex, a contradiction. Analogously, we reach a contradiction if \( v_i \) is a bad vertex. This completes the proof of Claim 9. (c)

We prove next that the graph \( G \) must contain a cut-vertex.

**Claim 10** The graph \( G \) contains a cut-vertex.

**Proof.** Suppose, to the contrary, that \( G \) is 2-connected. By Claim 9, all path-components of \( G[D] \) lie on a common cycle. Following the notation introduced in the proof of Claim 7 and Claim 9, let \( Q_1, \ldots, Q_\ell \) be the path-components of \( G[D] \). Further, let \( Q_i \) be a \((a_i, b_i)\)-path and let \( u_i \) and \( v_i \) be the two vertices of \( G - V(Q_i) \) associated with the path \( Q_i \) for \( i \in [\ell] \), where \( u_i \) is adjacent to \( a_i \) and \( v_i \) is adjacent to \( b_i \). Renaming components and vertices, if necessary, we may assume that the path-component follows each other in a natural order on \( C \), and so \( Q_{i+1} \) immediately follows \( Q_i \) on \( C \) for \( i \in [\ell] \setminus \{1\} \). Further, we may assume that \( C \) starts at \( u_1 \), proceeds along the edge \( u_1a_1 \), follows \( Q_1 \) from \( a_1 \) to \( b_1 \), and then proceeds along the edge \( b_1v_1 \) to \( v_1 \). Thereafter, we assume that \( u_{i+1} \) follows \( v_i \) for every \( i \in [\ell - 1] \) and every internal vertex of this \((v_i, u_{i+1})\)-subpath contains no vertex of degree 2.

By Claim 8, either all \( u_i \)'s are bad vertices and all \( v_i \)'s are good vertices or vice versa for \( i \in [\ell] \). We may assume that all \( u_i \)'s are bad vertices. Recall that by our earlier assumptions, \( G \) is not a cycle, and so \( \Delta(G) > 2 \). Let \( \mathcal{L} \) be the set of vertices of \( C \) that have degree at least 3 in \( G \). Since \( G \) is 2-connected, every vertex in \( \mathcal{L} \) is connected in \( G - E(C) \) to some other vertex in \( \mathcal{L} \). Among all paths in \( G - E(C) \) connecting two vertices of \( \mathcal{L} \), let \( P \) be a shortest such path. Let \( P \) be a \((w, z)\)-path, and so \( V(P) \cap V(C) = \{w, z\} \). We note that all vertices of \( P \) have degree at least 3 in \( G \).

Let \( Q_i \) be the component of \( G[D] \) that proceeds \( w \) on \( C \) and let \( Q_j \) be the component of \( G[D] \) that precedes \( z \) on \( C \) (in our chosen ordering of the vertices along \( C \)). Suppose first that \( v_i \) is a good vertex. Thus, all \( u_s \)'s are bad vertices and all \( v_s \)'s are good vertices for \( s \in [\ell] \). In particular, \( v_j \) is a good vertex. However, applying Claim 7 and Claim 8, traversing \( C \) from \( v_i \) to \( w \), then continuing along the path \( P \) to \( z \), and then following the path from \( z \) to \( v_j \) in \( C - v_i \), we deduce that \( v_j \) is a bad vertex, a contradiction. Analogously, we reach a contradiction if \( v_i \) is a bad vertex. This completes the proof of Claim 10. (c)

By Claim 10, the connected graph \( G \) contains a cut-vertex. We consider next the structure of the block graph of \( G \).
Claim 11 Every end-block of the block graph of $G$ is a cycle of length 3, 4 or 5.

Proof. Let $T_G$ denote the block graph of $G$, and let $C$ be an end-block of $G$. Further, let $w$ be the cut-vertex of $G$ that belongs to $C$. If $\delta(C) > 2$, then analogously as in the proof of Claim 2 we obtain an infinite process with infinite growth, noting that one branch of the infinite process may contain the vertex $w$ but this does not affect the infinite growth of the graph. Thus, $\delta(C) = 2$.

We show next that $C$ is a cycle. Suppose, to the contrary, that $\Delta(C) > 2$. If the vertex $w$ has neighbors in both $S_A$ and $S_B$ in the 2-connected graph $C$, then $C$ is a TDP-graph, and proceeding as in the proofs of Claims 7, 8, 9, and 10 we show that $C$ has a cut-vertex, a contradiction. Therefore, all neighbors of $w$ in $C$ have the same status. If $d_C(w) > 2$, then the vertex $w$ cannot belong to a path $Q$ and $R$ as defined in the proof of Claim 7, since then $w$ would have neighbors of both status in $C$, a contradiction. Hence, $d_C(w) = 2$. In this case, we simply choose the path-component $Q_1$ in $C$ to contain the vertex $w$, and using analogous arguments as in the proofs of Claims 7, 8, 9, and 10 we show that $C$ has a cut-vertex, a contradiction. Hence, $\Delta(C) = 2$, implying that $C$ is a cycle, as claimed.

Thus, $C \cong C_k$ for some $k \geq 3$. If $k \geq 7$, then we contradict our assumption that every path-component of $G[D]$ has order at most 4. Hence, $k \leq 6$. Suppose that $k = 6$. Let $C$ be the cycle $w_1w_2\ldots w_6w_1$, where $w = w_1$. Let $\text{sta}(w_1) = X$. Since $G$ is a TDP-graph, the vertex $w_2$ has a neighbor in both $S_A$ and $S_B$, implying that $\text{sta}(w_3) = \overline{X}$. Analogously, since vertex $w_6$ has a neighbor in both $S_A$ and $S_B$, we deduce that $\text{sta}(w_5) = \overline{X}$. But then both neighbors of $w_4$ have the same status, namely $\overline{X}$, a contradiction. Hence, $k \leq 5$. This completes the proof of Claim 11. (c)

Claim 12 If $G$ contains exactly one cut-vertex, then $(G, S)$ is the labeled base graph $G_3$.

Proof. Suppose that $G$ contains exactly one cut-vertex, say $v$. Thus, every block of $G$ is an end-block that contains the vertex $v$. Let $C$ be an arbitrary end-block of $G$. By Claim 11, the end-block $C$ is a cycle of length $k \in \{3, 4, 5\}$. Let $C$ be the cycle $v_1v_2\ldots v_kv$ and let $\text{sta}(v) = X$, where $k \in \{3, 4, 5\}$. If $k = 3$, then $\text{sta}(v_1) = \text{sta}(v_2) = \overline{X}$. If $k = 4$, then $\text{sta}(v_2) = \overline{X}$ and, renaming $v_1$ and $v_3$, if necessary, we may assume that $\text{sta}(v_1) = X$ and $\text{sta}(v_3) = \overline{X}$. If $k = 5$, then $\text{sta}(v_2) = \text{sta}(v_3) = \overline{X}$ and $\text{sta}(v_1) = \text{sta}(v_4) = X$. This implies that if $G$ contains an end-block that is a 4-cycle, then the vertex $v$ is a good cut-vertex, contradicting our earlier assumption that $G$ has no good cut-vertex. Hence, no end-block of $G$ is a 4-cycle. Thus, since $G$ is a TDP-graph, at least one end-block is a 3-cycle and at least one end-block is a 5-cycle. If $G$ contains at least three blocks, then the vertex $v$ is a good cut-vertex, a contradiction. Hence, $G$ contains exactly two end-blocks, one of length 3 and the other of length 5; that is, $(G, S)$ is the labeled base graph $G_3$. (c)

By Claim 12, we may assume that the connected graph $G$ contains at least two cut-vertices, for otherwise $(G, S) \in \mathcal{G}$. This implies that $G$ contains at least two end-blocks that are vertex disjoint. Among all cut-vertices that belong to an end-block, let $v$ and $v'$ be chosen to be at maximum distance apart in $G$. Let $C$ and $C'$ be an end-block containing $v$ and $v'$, respectively. We note that $v \neq v'$. 

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Claim 13 If $v$ and $v'$ are adjacent, then $(G, S)$ is the labeled base graph $G_2$ or $G_4$.

Proof. Suppose that $v$ and $v'$ are adjacent vertices. If there is a cut-vertex, $v''$ say, of $G$ distinct from $v$ and $v'$, then by our choice of $v$ and $v'$, the vertex $v''$ is adjacent to both $v$ and $v'$ and belongs to an end-block. Since $G$ is a sparse TDP-graph, two of the vertices in $\{v, v', v''\}$ have the same status and the third vertex from this set has a different status. However, both vertices of the same status from this set are good cut-vertices, a contradiction. Therefore, $v$ and $v'$ are the only cut-vertices in $G$. Let $\text{sta}(v) = X$. By Claim 11, $C \cong C_k$ and $C' \cong C_{k'}$ where $3 \leq k, k' \leq 5$.

If $k = 4$, then $v$ is a good cut-vertex, a contradiction. Hence, $k \in \{3, 5\}$. Analogously, $k' \in \{3, 5\}$. If $v$ belongs to two end-blocks, then $v$ is a good cut-vertex, a contradiction. Hence, $C$ is the only end-block containing $v$, implying that $d_G(v) = 3$. Analogously, $C'$ is the only end-block containing $v'$ and $d_G(v') = 3$. Thus, the graph $G$ is determined.

Suppose that $C \cong C_3$ and let $C$ be the cycle $v_1v_2v_3$, and so $\text{sta}(v_1) = \text{sta}(v_2) = X$ and $\text{sta}(v_3) = X$. If $C' \cong C_5$ and $C'$ is the cycle $v_1'v_2'v_3'v_4'v_5'$, then $\text{sta}(v_1') = \text{sta}(v_2') = X$ and $\text{sta}(v_3') = \text{sta}(v_4') = X$. But then all three neighbors of $v'$ have the same status, a contradiction. Hence, $C' \cong C_3$ and $C'$ is the cycle $v_1'v_2'v_3'$, then $\text{sta}(v_1') = \text{sta}(v_2') = X$, implying that $(G, S)$ is the labeled base graph $G_2$. Suppose, finally, that $C \cong C_5$ and $C' \cong C_5$. In this case, $(G, S)$ is the labeled base graph $G_4$. (c)

By Claim 13, we may assume that $d_G(v, v') \geq 2$, for otherwise $(G, S) \in G$, as desired. If $d_G(v) = 3$, then let $x$ be the unique neighbor of $v$ outside of $C$, otherwise, if $d_G(v) > 3$, let $x = v$. If $d_G(v') = 3$, then let $x'$ be the unique neighbor of $v$ outside of $C$, otherwise, if $d_G(v') > 3$, let $x' = v'$.

Claim 14 $x \neq x'$.

Proof. Suppose, to the contrary, that $x = x'$. This implies that $d_G(v) = d_G(v') = 3$ and $d_G(v, v') = 2$. Thus, $x$ is a cut-vertex in $G$. By assumption, no vertex in $D$ is a cut-vertex of $G$, and so $d_G(x) \geq 3$. Let $\text{sta}(v) = X$. Suppose firstly that $\text{sta}(v') = \overline{X}$. In this case, let $v''$ be an arbitrary neighbor of $x$ different from $v$ and $v'$. If $\text{sta}(v'') = X$, then $vx$ is a good bridge, while if $\text{sta}(v'') = \overline{X}$, then $v'x$ is a good bridge. Suppose secondly that $\text{sta}(v') = X$. In this case, noting that $x$ has a neighbor of status $\overline{X}$, both edges $vx$ and $v'x$ are good bridges in $G$. In all cases, we contradict our assumption that there is no good bridge in $G$. Therefore, $x \neq x'$. (c)

By Claim 14, $x \neq x'$. Let $L$ be a shortest $(v, v')$-path in $G$ and let $L'$ be the $(x, x')$-subpath of $L$. Possibly, $L = L'$. Suppose that every vertex of $L'$ has degree at least 3 in $G$. Proceeding as in the proofs of Claims 7, 8, 9, and 10, the vertex immediately preceding $x'$ on $L'$ has a different status to a neighbor of $x'$ not on $L'$. Therefore, $x'$ has a neighbor in $A$ and in $B$. If $x' \neq v'$, then this implies that $v'x'$ is a good bridge. If $x' = v'$, then this implies that $v'$ is a good cut-vertex. Both cases produce a contradiction. Therefore, at least one vertex of $L'$ has degree 2 in $G$.
Let $x_1$ be the first vertex on $L'$ of degree 2 in $G$, and let $P_1$ be the path-component of $G[D]$ containing $x_1$. If $x = x_1$, then this implies that $d_G(v) = 3$ and that $x$ is the unique neighbor of $v$ outside $C$. But then $x$ is a cut-vertex of $G$ of degree 2, a contradiction. Hence, $x \neq x_1$. Let $u_1$ be the vertex immediately preceding $x_1$ on $L'$, and let $z_1$ be the first vertex on $L'$ of degree at least 3 that follows $x_1$ on $L'$. Possibly, $x = u_1$.

Claim 15 $u_1$ is a good vertex associated with the path-component $P_1$.

**Proof.** Suppose that $x \neq u_1$. Since every vertex on the $(x, u_1)$-subpath of $L'$ have degree at least 3 in $G$, analogous arguments as before show that $u_1$ has a neighbor in $A$ and in $B$, and so $u_1$ is a good vertex associated with the path-component $P_1$. Suppose that $x = u_1$ and $x = v$. In this case, $d_G(v) \geq 4$. Since all neighbors of $v$ outside $C$ have the same status, this implies that $u_1$ is a good vertex associated with the path-component $P_1$. This in turn implies that $z_1$ is a bad vertex associated with the path-component $P_1$. Suppose that $x = u_1$ and $x \neq v$. In this case, $d_G(v) = 3$ and $vx$ is a bridge. Since $G$ contains no good bridge, this implies that all neighbors of $x$ in $G - V(C)$ have the same status. This in turn implies that $u_1$ is a good vertex associated with the path-component $P_1$. (c)

By Claim 15, $u_1$ is a good vertex associated with the path-component $P_1$. By assumption, $G$ contains no good path-component in $G[D]$. In particular, $P_1$ is not a good path-component, implying that $z_1$ is a bad vertex associated with the path-component $P_1$; that is, all neighbors of $z_1$ in $G - V(P_1)$ have the same status. We now consider the $(z_1, x')$-subpath, say $L_1$, of $L'$. If every vertex of $L_1$ has degree at least 3 in $G$, then analogously as in paragraph after the proof of Claim 14, $x'$ has a neighbor in $A$ and in $B$, producing a contradiction. Therefore, at least one vertex of $L_1$ has degree 2 in $G$.

Let $x_2$ be the first vertex on $L_1$ of degree 2 in $G$, and let $P_2$ be the path-component of $G[D]$ containing $x_2$. Let $u_2$ be the vertex immediately preceding $x_2$ on $L_1$, and let $z_2$ be the first vertex on $L_1$ of degree at least 3 that follows $x_2$ on $L_1$. Analogously as in the paragraph before Claim 15, we show that $u_2$ is a good vertex associated with the path-component $P_2$, and therefore that $z_1$ is a bad vertex associated with the path-component $P_2$. We now consider the $(z_2, x')$-subpath, say $L_2$, of $L'$. Continuing in this way, there exists some integer $t \geq 2$ such that the $(z_t, x')$-subpath, $L_t$ say, of $L'$ contains only vertices of degree at least 3 in $G$, implying that $x'$ has a neighbor in $A$ and in $B$, producing a contradiction. This completes the proof of Theorem 2. □

5 Closing Remarks

We close with a short discussion about the independence of operations $O_1$ to $O_{19}$ in the class $G$. For this purpose, we will construct small graphs in $G$ from our labeled base graphs that cannot be built by any other construction in $G$, thereby showing that operation $O_i$ is independent for each $i \in [19]$. The independence of these nineteen operations used to build graphs in the family $G$ show that none of them are redundant, and all are needed in the construction.
• Apply operation $O_2$ on $G_1$ (to obtain the graph $K_4 - e$).

• Apply operation $O_3$ on $G_1$ to obtain the house graph; that is, the graph obtained from a 5-cycle by adding an edge.

• Apply operation $O_1$ once and operation $O_2$ three times on the house graph, to obtain $K_5$.

• Apply operation $O_4$ to two nonadjacent vertices of degree 2 on $G_2$.

• The independence of operation $O_x$, where $x \in \{5, 6, 11, 12, 13, 14, 15, 16\}$, can be seen by applying $O_x$ once on $G_1$.

• The independence of operation $O_x$, where $x \in \{7, 10, 18\}$, can be seen by applying $O_x$ once on adjacent vertices of degree 3 in $G_2$.

• The independence of operation $O_x$, where $x \in \{8, 9, 19\}$, can be seen by applying $O_x$ once on adjacent vertices of degree 3 in $G_4$.

• Apply operation $O_{17}$ once on the cut-vertex of $G_3$.

References


