Abstract

Let $G = (V, E)$ be a graph of order $n$. A closed distance magic labeling of $G$ is a bijection $\ell : V(G) \rightarrow \{1, \ldots, n\}$ for which there exists a positive integer $k$ such that $\sum_{x \in N[v]} \ell(x) = k$ for all $v \in V$, where $N[v]$ is the closed neighborhood of $v$. We consider the closed distance magic graphs in the algebraic context. In particular we analyze the relations between the closed distance magic labelings and the spectra of graphs. These results are then applied to the strong product of graphs with complete graph or cycle and to the circulant graphs. We end with a number theoretic problem whose solution results in another family of closed distance magic graphs somewhat related to the strong product.

Keywords: distance magic graphs, graph spectrum, strong product

AMS subject classification (2010): 05C78, 05C50, 05C76
1 Introduction and preliminaries

All graphs considered in this paper are simple finite graphs. For a graph \( G \), we use \( V(G) \) for the vertex set and \( E(G) \) for the edge set of \( G \). The open neighborhood \( N(x) \) (or more precisely \( N_G(x) \), when needed) of a vertex \( x \) is the set of all vertices adjacent to \( x \), and the degree \( d(x) \) of \( x \) is \( |N(x)| \), i.e. the size of the neighborhood of \( x \). With \( N[x] \) (or \( N_G[x] \)) we denote the closed neighborhood \( N(x) \cup \{ x \} \) of \( x \). By \( C_n \) we denote a cycle on \( n \) vertices.

Different kinds of labelings have been important part of graph theory by now. See a dynamic survey [10] which covers the field. One type of labelings includes the magic labelings, where some objects (edges, vertices, etc.) of a graph must be labeled in such a way, that certain sums (depending on graph properties) are constant. Closed distance magic labeling (also called \( \Sigma' \)-labeling, see [3]) of a graph \( G = (V(G), E(G)) \) of order \( n \) is a bijection \( \ell \colon V(G) \to \{1, \ldots, n\} \) with the property that there is a positive integer \( k' \) (called the magic constant) such that \( w(x) = \sum_{y \in N_G[x]} \ell(y) = k' \) for every \( x \in V(G) \), where \( w(x) \) is the weight of \( x \). If a graph \( G \) admits a closed distance magic labeling, then we say that \( G \) is closed distance magic graph.

Closed distance magic graphs are analogue to distance magic graphs, where the sums are taken over the open neighborhoods \( N_G(x) \) instead of the closed ones \( N_G[x] \), see [2, 7, 8].

Let \( D \) be a subset of non-negative integers. A O’Neal and Slater in [16] have defined the \( D \)-distance magic labeling as a bijection \( f : V(G) \to \{1, \ldots, n\} \) such that there is a magic constant \( k \) such that for any vertex \( x \in V(G) \), \( w(x) = \sum_{y \in N_D(x)} f(y) = k \). Here \( N_D(x) = \{ y \in V(G) | d(x, y) \in D \} \), i.e. the weight of a vertex \( x \in V(G) \) is the sum of the labels of all the vertices \( y \in V(G) \) for which their distance to \( x \) belongs to \( D \). This kind of labeling has been studied e.g. by Simanjuntak et al. in [17] (we refer to some of their results in one of the following sections). This kind of labeling is a generalization of both distance magic labeling and closed distance magic labeling, where \( D = \{1\} \) and \( D = \{0, 1\} \), respectively.

The concept of distance magic labeling has been motivated by the construction of magic rectangles. Magic rectangles are natural generalization of the magic squares that has been intriguing mathematicians and the general public for a long time [11]. A magic \((m, n)\)-rectangle \( S \) is an \( m \times n \) array in which the first \( mn \) positive integers are placed so that the sum over each row of \( S \) is constant and the sum over each column of \( S \) is another (different
if $m \neq n$) constant. Harmuth proved that:

**Theorem 1.1** ([13, 14]) *For $m, n > 1$ there is a magic $(m, n)$-rectangle $S$ if and only if $m \equiv n \pmod{2}$ and $(m, n) \neq (2, 2)$.*

A related concept is the notion of distance antimagic labeling. This is again a bijection $f$ from $V(G)$ to $\{1, \ldots, n\}$ but this time different vertices are required to have distinct weights (where the sums are taken over the open neighborhoods $N_G(x)$). A more restrictive version of this labeling is the $(a, d)$-distance antimagic labeling. It is a distance antimagic labeling with the additional property that the weights of vertices form an arithmetic progression with difference $d$ and first term $a$. If $d = 1$, then $f$ is called simply distance antimagic labeling [9]. Notice that if a graph has a closed distance magic labeling then it has a distance antimagic labeling. The opposite is however not true, as it can be easily checked on the example of $C_5$. It has no closed distance magic labeling, while it has a distance antimagic labeling $f(v_i) = i$, where $V(C_5) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(C_5) = \{{v_i, v_j}\} : |i - j| = 1 \lor |i - j| = 4$.

Finding an $r$-regular distance antimagic labeling turns out equivalent to finding a fair incomplete tournament $FIT(n, r)$ [9]. A fair incomplete tournament of $n$ teams with $g$ rounds, $FIT(n, r)$, is a tournament in which every team plays $r$ other teams and the total strength of the opponents that team $i$ plays is $S_{n,r}(i) = (n+1)(n-2)/2 + i - c$ for every $i$ and some fixed constant $c$.

We recall one out of four standard graph products (see [12]). Let $G$ and $H$ be two graphs. The strong product $G \Box H$ is a graph with vertex set $V(G) \times V(H)$. Two vertices $(g, h)$ and $(g', h')$ are adjacent in $G \Box H$ if either $g = g'$ and $h$ is adjacent with $h'$ in $H$, or $h = h'$ and $g$ is adjacent with $g'$ in $G$, or $g$ is adjacent with $g'$ in $G$ and $h$ is adjacent with $h'$ in $H$. Recently in [1] two other standard products, namely direct and lexicographic, have been considered with respect to the property of being distance magic.

It is easy to notice the following observation, that will be useful in our further considerations.

**Observation 1.2** If $G$ is an $r$-regular closed distance magic graph on $n$ vertices, then $k' = \frac{(r+1)(n+1)}{2}$.

In the next section we reveal somewhat surprising connection between the existence of closed distance magic labeling and the spectrum of a graph. In
the following sections we consider the existence of closed distance magic labelings of chosen families of graphs using algebraic tools developed in Section 2. In the final section we present a combinatorial problem whose solution yields more closed distance magic graphs.

2 Necessary conditions - algebraic approach

Let \( V(G) = \{x_1, \ldots, x_n\} \). Then the following system of equations with unknowns \( \ell(x_1), \ldots, \ell(x_n) \) has to be satisfied for every closed distance magic graph:

\[
\begin{align*}
w(x_1) &= k', \\
w(x_2) &= k', \\
&\vdots \\
w(x_n) &= k'.
\end{align*}
\]

(1)

By writing this system in matrix form, we get

\[
(A(G) + I_n)l = k'u_n,
\]

where \( A(G) \) is the adjacency matrix of \( G \), \( I_n \) is \( n \times n \) identity matrix, \( l = (\ell(x_1), \ldots, \ell(x_n)) \) and \( u_n \) is a vector of length \( n \) with every entry equal to 1.

It is well known that the rank of a square matrix is equal to the number of its non-zero singular values (see e.g. [6], p.31). In the case of symmetric matrices it is in turn equal to the number of the non-zero eigenvalues. It means that the dimension of the set of solutions of the system (1) equals to the multiplicity of \( 0 \) in the spectrum of \( A(G) + I_n \), i.e. to the multiplicity of \( -1 \) in \( Sp(G) \), where \( Sp(G) \) denotes the spectrum of \( G \).

It has been recently proved by O’Neil and Slater in [15] that if a graph is closed distance magic, then the magic constant \( k' \) is unique, i.e. even if there exist two distinct closed distance magic labelings, then they result in same magic constant \( k' \).

The above considerations lead us to the following result.

**Theorem 2.1** If \( G \) is a closed distance magic graph and the system (1) has \( k + 1 \) linearly independent solutions, then the multiplicity of \( -1 \) in \( Sp(G) \) is \( k \).

The following corollary will be used in the remainder of the paper.
Corollary 2.2 Let $G$ be a closed distance magic graph such that there exist $k$ linearly independent solutions to the system (1) such that no bijection $\ell: V(G) \to \{1, \ldots, |V(G)|\}$ is their linear combination. Then the multiplicity of $-1$ in $Sp(G)$ is at least $k$.

If $G$ is an $r$-regular graph, then clearly the system (1) has at least one solution not being bijection from $V(G)$ to $\{1, \ldots, |V(G)|\}$, namely $\ell(x_1) = \cdots = \ell(x_n) = k'/(r+1)$. Hence the following holds:

Corollary 2.3 If $G$ is a regular closed distance magic graph, then $-1 \in Sp(G)$.

Perfect code is a subset $C(G)$ of $V(G)$ such that the closed neighborhoods of the vertices $v \in C$ form the partition of $G$. It is known that every graph $G$ with a perfect code must satisfy $-1 \in Sp(G)$ (see e.g. [4], p. 22). Observe however that in the case of a regular graph $G$ having a perfect code, the system (1) has at least the following two solutions: $\ell(x_1) = \cdots = \ell(x_n) = k'/r + 1$ and $\ell(x) = k', x \in C(G)$, $\ell(x) = 0, x \in V(G) - C(G)$. Obviously these solutions are linearly independent and no bijection $\ell: V(G) \to \{1, \ldots, |V(G)|\}$ is their linear combination. This means that the following is true.

Corollary 2.4 Let $G$ be a regular closed distance magic graph having a perfect code. Then the multiplicity of $-1$ in $Sp(G)$ is at least 2.

The following fact can be found e.g. in [5], p.11.

Fact 2.5 Given any graphs $G$ and $H$, the eigenvalues of $G \boxtimes H$ have the form $(\lambda_G + 1)(\lambda_H + 1) - 1$, where $\lambda_G \in Sp(G)$ and $\lambda_H \in Sp(H)$.

It follows that the value $-1$ can appear in $Sp(G \boxtimes H)$ if and only if it appears in the spectrum of at least one of the graphs $G$ and $H$. This leads us to the following corollary.

Corollary 2.6 If graphs $G$ and $H$ are regular and $G \boxtimes H$ is closed distance magic, then $-1 \in Sp(G) \cup Sp(H)$.

If $G$ is an $r$-regular graph with $s$ distinct eigenvalues $r, \lambda_1, \ldots, \lambda_{s-1}$, then its line graph $L(G)$ has at most $s + 1$ distinct eigenvalues $2r - 2, r + \lambda_1 - 2, \ldots, r + \lambda_{s-1} - 2, -2$. So the following is true.
Corollary 2.7 Let $G$ be an $r$-regular graph, $r > 1$. If its line graph $L(G)$ is closed distance magic, then $1 - r \in Sp(G)$. ■

In the following sections we are going to discuss the existence of closed distance magic labelings of chosen families of graphs.

3 Complete graphs and their strong products

It is obvious that every complete graph is closed distance magic (observe that every bijection $\ell : V(G) = K_n \to \{1, \ldots, n\}$ results in equal vertex weights). This is however not true in the case of the complete bipartite graphs. Corollary 2.3 may be generalized on some non-regular graphs. It is enough that we are able to find any labeling satisfying the system of equations (1), not being the bijection from $V(G)$ to $\{1, \ldots, n\}$. Thus, for example, the statement in Corollary 2.3 remains true also for complete bipartite graphs $K_{m,n}$, where $2 \leq m < n$. Here for $V(K_{m,n}) = \{x_1, \ldots, x_n\} \cup \{x_{n+1}, \ldots, x_{n+m}\}$, the sample solution of the system (1) is $\ell(x_1) = \cdots = \ell(x_n) = k'/m$ and $\ell(x_{n+1}) = \cdots = \ell(x_{n+m}) = k'/n$. This leads us immediately to the following result.

Proposition 3.1 For any $m$ and $n$ such that $2 \leq m \leq n$, $K_{m,n}$ is not closed distance magic.

Proof. The spectrum of the graph $K_{m,n}$ is $Sp(K_{m,n}) = \{-\sqrt{mn}, 0^{m+n-2}, \sqrt{mn}\}$ (see [5], p.8.), thus $-1$ is never its element for $m, n \geq 2$. In consequence, for any value $k'$, the only solution of the system (1) is $\ell(x_1) = \cdots = \ell(x_n) = k'/m$ and $\ell(x_{n+1}) = \cdots = \ell(x_{n+m}) = k'/n$. ■

The following result related with strong product has been proved by Beena.

Theorem 3.2 ([3]) Let $G$ be any connected $r$-regular graph, $r > 0$. If $n$ is even, then $K_n \boxtimes G$ is a closed distance magic graph. ■

Next result extends the Theorem 3.2 for the case when both graphs have odd order.

Theorem 3.3 Let $n$ be an odd number and $r > 0$. If $G$ is an $r$-regular graph of odd order, then $K_n \boxtimes G$ is a closed distance magic graph.
Proof. Let \( V(K_n) = \{y_0, \ldots, y_{n-1}\} \) and \( V(G) = \{x_0, \ldots, x_{m-1}\} \) for odd \( m \) and \( n \). For a vertex \((y_i, x_j)\) from \( V(K_n \boxtimes G) \) we simply write \( v_{i,j} \). Notice that if \( x_p x_q \in E(G) \), then \( v_{i,q} \in N_{H}(v_{i,p}) \) for every \( j \in \{0, \ldots, n-1\} \). There exists a magic \((n, m)\)-rectangle by Theorem 1.1. Let \( a_{i,j} \) be an \((i, j)\)-entry of the \((n, m)\)-rectangle, \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). Define the labeling \( \ell : V(K_n \boxtimes G) \to \{1, \ldots, nm\} \) as \( \ell(v_{i,j}) = a_{i+1,j+1} \) for \( i \in \{0, \ldots, n-1\} \) and \( j \in \{0, \ldots, m-1\} \). Obviously \( \ell \) is a bijection and moreover for each \( j \) we have \( \sum_{i=0}^{n-1} \ell(v_{i,j}) = C \) for some constant \( C \). Therefore for any \( x \in V(K_n \boxtimes G) \) we have \( w(x) = (r + 1)C = k' \).

An \( r \)-regular graph \( G \) is called \((r, a, b)\)-strongly regular if every pair of adjacent vertices has \( a \geq 0 \) common neighbors and every pair of non-adjacent vertices has \( b \geq 1 \) common neighbors. The following is true.

**Proposition 3.4** If a strongly regular graph \( G \) on \( n \) vertices is closed distance magic, then \( G \cong K_n \).

**Proof.** The eigenvalues of \( G \) are \( r \) and two roots \( x_1, x_2 \) of the equation \( x^2 + (c - a)x + c - r = 0 \) (see e.g. [4], p. 20). By the Corollary 2.3, \(-1\) must be solution of this equation and thus \( a = r - 1 \). This implies that \( G \) is disjoint union of some number of copies of \( K_n \). In such a graph, the non-adjacent vertices do not have common neighbors. As in the strongly regular graph every pair of non-adjacent vertices has at least one vertex in common, we have \( G \cong K_n \).

### 4 Cycles and their strong products

It is very easy to see that directly that \( C_3 \) is the only closed distance magic graph among cycles. Nevertheless, we start this section with (also easy) algebraic proof of this fact in order to present a simple application of the methods developed in Section 2. For this we need the following fact from e.g. in [5], p.9.

**Fact 4.1** The spectrum of an undirected cycle \( C_n \) consists of the numbers \( 2 \cos(2\pi j/n), j \in \{1, \ldots, n\} \).

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If \(-1 \in \text{Sp}(C_n)\), then we have for some \(j\)

\[
2 \cos(2\pi j/n) = -1,
\]
what is equivalent to the fact that \(j \in \{n/3, 2n/3\}\). It implies in turn that \(n \equiv 0 \pmod{3}\). Moreover, as \(j\) can take one of two values, it means that in this case the multiplicity of \(-1\) in \(\text{Sp}(C_n)\) is exactly 2. This leads us to the following results.

**Proposition 4.2** The only closed distance magic cycle is \(C_3 = K_3\).

**Proof.** Assume \(C_n\) is closed distance magic with magic constant \(k'\) and denote its vertices by \(x_0, x_1, \ldots, x_{n-1}\). We can assume that \(n \equiv 0 \pmod{3}\).

Let us consider the following three labelings of \(V(G)\). For every \(i \in \{0, 1, 2\}\) and \(j \in \{0, \ldots, n-1\}\) set

\[
\ell_i(x_j) = \begin{cases} 
  k', & \text{if } j \equiv i \pmod{3}, \\
  0, & \text{if } j \not\equiv i \pmod{3}.
\end{cases}
\]

Obviously these labelings satisfy the system (1) and are linearly independent. Moreover for any linear combination \(\ell\) of \(\ell_0, \ell_1\) and \(\ell_2\), we have \(\ell(x_{j_1}) = \ell(x_{j_2})\) for \(j_1 \equiv j_2 \pmod{3}\), so there is no bijection \(\ell: V(G) \to \{1, \ldots, n\}\) linearly independent from \(\ell_0, \ell_1\) and \(\ell_2\) if \(n > 3\). As the multiplicity of \(-1\) in \(\text{Sp}(C_n)\) is 2, the cycle is not closed distance magic if \(n > 3\).

The following result is a direct consequence of Corollary 2.2 and the discussion before Proposition 4.2.

**Corollary 4.3** If the product \(G \times C_n\) is closed distance magic, where \(G\) is a regular graph, then \(n \equiv 0 \pmod{3}\) or \(-1 \in \text{Sp}(G)\).

**Corollary 4.4** Let the product \(G \times C_n\) be closed distance magic, where \(G\) is a \(2d\)-regular graph on \(m\) vertices. Then \(n \equiv 1 \pmod{2}\) and \(m \equiv 1 \pmod{2}\). Moreover, \(n \equiv 3 \pmod{6}\) or \(-1 \in \text{Sp}(G)\).

**Proof.** Let \(k'\) be the magic constant of \(G \times C_n\). The strong product of \(r_1\)- and \(r_2\)-regular graphs is \(((r_1 + 1)(r_2 + 1) - 1)\)-regular, so by the Observation 1.2 we have

\[
k' = 3(2d + 1)(mn + 1)/2.
\]
This means that $m \equiv 1 \pmod{2}$ and $n \equiv 1 \pmod{2}$. Now if $-1 \not\in Sp(G)$, then by the Corollary 4.3, $n \equiv 0 \pmod{3}$ and in consequence $n \equiv 3 \pmod{6}$.

Let $V(C_m \boxtimes C_n) = \{v_{ij} : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$, where $N(v_{ij}) = \{v_{i-1,j-1}, v_{i-1,j+1}, v_{i,j}, v_{i,j+1}, v_{i+1,j-1}, v_{i+1,j+1}\}$ and operation on the first suffix is taken modulo $m$ and on the second suffix modulo $n$. We also refer to the set of all vertices $v_{ij}$ with fixed $i$ as $i$-th row and with fixed $j$ as $j$-th column. Below we give the necessary and sufficient conditions for $C_m \boxtimes C_n$ to be closed distance magic.

**Theorem 4.5** The strong product $C_m \boxtimes C_n$ is closed distance magic if and only if at least one of the following conditions holds:

1. $m \equiv 3 \pmod{6}$ and $n \equiv 3 \pmod{6}$.
2. $\{m, n\} = \{3, x\}$ and $x$ is an odd number.

*Proof.* The following Lemma gives the necessary conditions for $C_m \boxtimes C_n$ to be closed distance magic.

**Lemma 4.6** If $C_m \boxtimes C_n$ is a closed distance magic graph, then at least one of the following conditions holds:

1. $m \equiv 3 \pmod{6}$ and $n \equiv 3 \pmod{6}$.
2. $\{m, n\} = \{3, x\}$ and $x$ is an odd number.

*Proof.* Assume that $C_m \boxtimes C_n$ is a closed distance magic graph with some magic constant $k'$. Hence there exists a closed distance magic labeling $\ell : V(C_m \boxtimes C_n) \to \{1, \ldots, mn\}$. By Corollary 4.4, $m$ and $n$ are odd. Assume that $m \not\equiv 3 \pmod{6}$.

Let us consider the following $2m+1$ solutions of the system $(1)$. For $i \in \{0, \ldots, m-1\}, j \in \{0, \ldots, n-1\}$ and $s \in \{1, \ldots, m\}$ set.

$$
\ell_s(x_{ij}) = \begin{cases} 
k'/3 & \text{if } i = s \land j \equiv 0 \pmod{3}, 
k'/3 & \text{if } i \neq s \land j \equiv 1 \pmod{3}, 
0 & \text{otherwise,}
\end{cases}
$$

$$
\ell_{m+s}(x_{ij}) = \begin{cases} 
k'/3 & \text{if } i = s \land j \equiv 1 \pmod{3}, 
k'/3 & \text{if } i \neq s \land j \equiv 2 \pmod{3}, 
0 & \text{otherwise,}
\end{cases}
$$
\[ \ell_{2m+1}(x_{ij}) = \begin{cases} \frac{k'}{3} & \text{if } j \equiv 0 \pmod{3}, \\ 0 & \text{otherwise.} \end{cases} \]

The first 2m solutions are linearly independent as the value of \( \ell(x_{ij}) \) for \( i \in \{0, \ldots, m\}, j \equiv x \pmod{3}, x \in \{0,1\} \), is not equal to 0 only in the solution \( \ell_{i+(j \mod 3)m} \). The last solution is not a linear combination of the other ones as in order to obtain \( \ell(x_{ij}) = k'/3 \) for \( j \equiv 0 \pmod{3} \) and \( \ell(x_{ij}) = 0 \) for \( j \equiv 1 \pmod{3} \) we should have \( \ell_{2m+1}(x_{ij}) = \sum_{i=1}^{m} (\ell_i(x) - (m-1)\ell_{m+i}(x)) \) for every \( x \), which obviously does not result with \( \ell(x_{ij}) = 0 \) for \( j \equiv 2 \pmod{3} \).

Moreover, no bijection \( \ell: V(G) \to \{1, \ldots, n\} \) can be a linear combination of \( \ell_i, i \in \{1, \ldots, 2m+1\} \) for \( n > 3 \) as for any \( \alpha_1, \ldots, \alpha_{2m+1} \) we have \( \sum_{i=1}^{2m+1} \alpha_i \ell_i(x_{ij}) = \sum_{i=1}^{2m+1} \alpha_i \ell_i(x_{ij}) \) for \( j_1 \equiv j_2 \pmod{3} \). As the multiplicity of \(-1\) in \( Sp(G) \) is \( 2m \), \( G = C_m \boxtimes C_n \) is not closed distance magic. This implies that \( n = 3 \). Moreover, \( m \) is an odd number as noted before.

The sufficiency of the second condition follows from the Theorem 3.3, as \( C_3 \cong K_3 \). In order to finish the proof of the Theorem we are going to show that there exists a closed distance magic labeling of \( C_m \boxtimes C_n \) for any \( m \equiv 3 \pmod{6} \) and \( n \equiv 3 \pmod{6} \).

First let us observe the following.

**Fact 4.7** For every \( m \equiv 3 \pmod{6} \) one can divide the set \( \{1, \ldots, m\} \) into \( m/3 \) mutually disjoint triples such that the sum of the elements of each triple equals to \( 3(m+1)/2 \).

**Proof.** The desired triples are \((2i+1, \frac{m+1}{2} - i, m - i)\) for \( i \in \{0, \ldots, \frac{m-3}{6}\} \) and \((2i+2, \frac{2m}{3} - i, \frac{5m-3}{6} - i)\) for \( i \in \{0, \ldots, \frac{m-9}{6}\} \). ■

Let us prove the existence of the desired labeling.

**Lemma 4.8** If \( m \equiv 3 \pmod{6} \) and \( n \equiv 3 \pmod{6} \), then \( C_m \boxtimes C_n \) is a closed distance magic graph.

**Proof.** Let us denote the triples granted by the Fact 4.7 for the set \( \{1, \ldots, m\} \) by \( S_0, \ldots, S_{m/3-1} \) and the elements of a triple \( S_p, p \in \{0, \ldots, m/3 - 1\} \), by \( s^0_p, s^1_p \) and \( s^2_p \). Similarly, let the triples for the set \( \{1, \ldots, n\} \) be \( T_0, \ldots, T_{n/3-1} \) and denote the elements of a triple \( T_q, q \in \{0, \ldots, n/3 - 1\} \), by \( t^0_q, t^1_q \) and \( t^2_q \).
Let us define two following labellings of \( C_m \oplus C_n \):

\[
\ell_1(v_{ij}) = s^j_{i \lfloor i/3 \rfloor} \mod 3, i \in \{0, \ldots, m-1\}, j \in \{0, \ldots, n-1\},
\]

\[
\ell_2(v_{ij}) = t^i_{j \lfloor j/3 \rfloor} \mod 3, i \in \{0, \ldots, m-1\}, j \in \{0, \ldots, n-1\}.
\]

In other words in order to construct the labeling \( \ell_1 \) we put the elements of \( S_0 \) on consecutive triples of vertices in the row 0, and then we label rows 1 and 2 in the same way. To label rows 3, 4 and 5 we use \( S_1 \), next three rows we label with \( S_2 \) and so on. Similarly the triples \( T_0, T_1, \ldots \) are used to label the columns of \( C_m \oplus C_n \). As an example, below we present the labellings \( \ell_1 \) and \( \ell_2 \) in the case of \( C_{15} \oplus C_9 \). In this case the partition of \( \{1, 2, \ldots, 15\} \) is \( (1, 8, 15), (3, 7, 14), (5, 6, 13), (2, 10, 12), (4, 9, 11) \) and the partition of \( \{1, 2, \ldots, 9\} \) is \( (1, 5, 9), (3, 4, 8), (2, 6, 7) \).

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Observe that for every two vertices \( v_{i_1j_1} \neq v_{i_2j_2} \), we have \( (\ell_1(v_{i_1j_1}), \ell_2(v_{i_1j_1})) \neq (\ell_1(v_{i_2j_2}), \ell_2(v_{i_2j_2})) \). Indeed, if \( (\ell_1(v_{i_1j_1}), \ell_2(v_{i_1j_1})) = (\ell_1(v_{i_2j_2}), \ell_2(v_{i_2j_2})) \), then we have \( s^j_{i_1 \lfloor i_1/3 \rfloor} \mod 3 = s^j_{i_2 \lfloor i_2/3 \rfloor} \mod 3 \) and \( t^i_{j_1 \lfloor j_1/3 \rfloor} \mod 3 = t^i_{j_2 \lfloor j_2/3 \rfloor} \mod 3 \). This implies in turn that \( \lfloor i_1/3 \rfloor = \lfloor i_2/3 \rfloor \) and \( i_1 \) (mod 3) = \( i_2 \) (mod 3) and finally \( i_1 = i_2 \). Similarly we can deduce that \( j_1 = j_2 \).
Let us define the labeling \( \ell : V(C_m \boxtimes C_n) \to \{1, \ldots, mn\} \) as
\[
\ell(x) = (\ell_1(x) - 1)n + \ell_2(x).
\]

It is straightforward to see that \( \ell \) is a bijection. The labeling \( \ell \) of \( C_{15} \boxtimes C_9 \) is given below.

\[
\begin{array}{cccccccccccc}
  \ell & 1 & 64 & 127 & 3 & 66 & 129 & 2 & 65 & 128 \\
 5 & 68 & 131 & 4 & 67 & 130 & 6 & 69 & 132 \\
 9 & 72 & 135 & 8 & 71 & 134 & 7 & 70 & 133 \\
 19 & 55 & 118 & 21 & 57 & 120 & 20 & 56 & 119 \\
 23 & 59 & 122 & 22 & 58 & 121 & 24 & 60 & 123 \\
 27 & 63 & 126 & 26 & 62 & 125 & 25 & 61 & 124 \\
 37 & 46 & 109 & 39 & 48 & 111 & 38 & 47 & 110 \\
 41 & 50 & 113 & 40 & 49 & 112 & 42 & 51 & 114 \\
 45 & 54 & 117 & 44 & 53 & 116 & 43 & 52 & 115 \\
 10 & 82 & 100 & 12 & 84 & 102 & 11 & 83 & 101 \\
 14 & 86 & 104 & 13 & 85 & 103 & 15 & 87 & 105 \\
 18 & 90 & 108 & 17 & 89 & 107 & 16 & 88 & 106 \\
 28 & 73 & 91 & 30 & 75 & 93 & 29 & 74 & 92 \\
 32 & 77 & 95 & 31 & 76 & 94 & 33 & 78 & 96 \\
 36 & 81 & 99 & 35 & 80 & 98 & 34 & 79 & 97 \\
\end{array}
\]

On the other hand observe that for every \( x \in V(C_m \boxtimes C_n) \) we have
\[
\sum_{y \in N[x]} \ell_1(y) = 3(s_0^0 + s_1^1 + s_2^2) = 9(m + 1)/2
\]
and
\[
\sum_{y \in N[x]} \ell_2(y) = 3(t_0^0 + t_1^1 + t_2^2) = 9(n + 1)/2.
\]

It means that for every \( x \in V(C_m \boxtimes C_n) \)
\[
\sum_{y \in N[x]} \ell(y) = \sum_{y \in N[x]} (\ell_1(y) - 1)n + \ell_2(y)
= n \sum_{y \in N[x]} \ell_1(y) - n(d(x) + 1) + \sum_{y \in N[x]} \ell_2(y)
= 9n(m + 1)/2 - 9n + 9(n + 1)/2 = 9(mn + 1)/2 = k',
\]
so \( \ell \) is closed distance magic labeling of \( C_m \boxtimes C_n \).

Clearly the last lemma finishes the proof of Theorem 4.5.
5 Circulant graphs

The circulant graph $Ci(n, S)$, where $S \subset \{1, \ldots, \lfloor n/2 \rfloor \}$ is the graph with vertex set $\{v_0, \ldots, v_{n-1}\}$, where $v_i$ and $v_j$ are adjacent if and only if $|i - j| \in S$. Observe that each closed distance magic labeling of $Ci(n, S)$ is exactly a $D$-distance magic labeling of a cycle $C_n$, where $D = S \cup \{0\}$. In particular, in [17] Simanjuntak et al. have proved two following facts.

**Proposition 5.1 ([17], Corollary 2)** For a positive integer $k$, the circulant graph $Ci(n, \{1, \ldots, k-1, k+1, \ldots, \lfloor n/2 \rfloor \})$ is closed distance magic if and only if $n = 4k$.

**Proposition 5.2 ([17], Theorem 6)** For $n \geq 2k + 2$ the circulant graph $Ci(n, \{1, \ldots, k\})$ is not closed distance magic.

We generalize the last result by the following observation.

**Lemma 5.3** Let $G = Ci(n, \{c, 2c, \ldots, kc\})$, where $k \geq 1$. If $G$ is closed distance magic, then $n = 2kc$ or $n = (2k + 1)c$.

*Proof.* Suppose that the graph $G$ is closed distance magic with the closed distance magic labeling $\ell$, then we obtain that $w(v_{kc}) - w(v_{(k+1)c}) = 0$. Hence for $n > 2kc$ we obtain $\ell(v_0) - \ell(v_{(2k+1)c}) = 0$, where the operation on the suffix is taken modulo $n$. If $n > 2kc$, this implies immediately that $(2k + 1)c \equiv 0 \pmod{n}$ and in consequence $n = (2k + 1)c$. On the other hand, by definition of $G$ we have $n \geq 2kc$ and the thesis follows.

Two following observations give the sufficient conditions for the existence of a closed distance magic labeling of $G = Ci(n, \{c, 2c, \ldots, kc\})$.

**Lemma 5.4** Let $G = Ci(n, \{c, 2c, \ldots, kc\})$. If $n = 2kc$ then $G$ is closed distance magic.

*Proof.* Let $\ell(v_i) = i + 1$, $\ell(v_{ck+i}) = n - i$ for $i \in \{0, 1, \ldots, ck - 1\}$. Obviously $\ell$ is a bijection and moreover for each $j \in \{0, \ldots, n - 1\}$ we have $w(x_j) = k(n + 1)$.

**Lemma 5.5** Let $G = Ci(n, \{c, 2c, \ldots, kc\})$ and $n = (2k + 1)c$. $G$ is closed distance magic if and only if $c$ is odd.
Proof. In the case when \( c = 1 \), \( G \) is a complete graph and thus also closed distance magic graph. Hence we can focus on the case when \( c \geq 2 \). Notice that \( G \) is \( 2k \)-regular. By Observation 1.2 there is no closed distance magic \( 2k \)-regular graph \( G \) with \( n \) being even. Thus \( c \) has to be odd.

Conversely, if \( c \) is an odd integer, then there exists a magic \( (2k + 1, c) \)-rectangle by Theorem 1.1. Let \( a_{i,j} \) be an \((i, j)\)-entry of the \((2k + 1, c)\)-rectangle, \( 1 \leq i \leq 2k + 1 \) and \( 1 \leq j \leq c \). Define the labeling \( \ell : V(G) \to \{1, \ldots, n\} \) as \( \ell(v_{ic+j}) = a_{i+1,j+1} \) for \( i \in \{0, \ldots, 2k\} \) and \( j \in \{0, \ldots, c-1\} \). Obviously \( \ell \) is a bijection and moreover for each \( t \in \{0, 1, \ldots, n-1\} \) we have \( w(v_t) = k' \).

As an immediate consequence of the Lemmas 5.3–5.5 we obtain the following theorem:

**Theorem 5.6** Let \( G = Ci(n, \{c, 2c, \ldots, kc\}) \). The graph \( G \) is closed distance magic if and only if either \( n = 2kc \) or \( n = (2k+1)c \) and \( c \) is odd.

Observe that if \( n \equiv 0 \pmod{c} \), then \( Ci(2kc, \{c, 2c, \ldots, kc\}) \cong cK_{2k} \) and \( Ci((2k+1)c, \{c, 2c, \ldots, kc\}) \cong cK_{2k+1} \). Thus we obtain the following corollary.

**Corollary 5.7** Given \( n \geq 2 \) and \( c \geq 1 \), the union \( cK_n \) is closed distance magic if and only if \( n(c+1) \equiv 0 \pmod{2} \).

We finish this section with necessary conditions for \( Ci(n, \{1, 2, \ldots, k-1, k+1\}) \) to be closed distance magic. We know that the set of distinct eigenvalues of \( G = Ci(n, S) \) is \( \{\lambda_j = \sum_{s=2}^{n} a_s \omega^{(s-1)j}, j \in \{0, \ldots, n-1\}\} \), where \( \omega = \exp(2\pi i/n) \) and \( a_s \) is the \( s \)-th entry of the first row of the adjacency matrix of \( G \) (see e.g. [4], p. 16). This can be rewritten as \( \{\lambda_j = 2 \sum_{s \in S} \cos(2j\pi s/n)\} \). Before we proceed, let us prove the following facts.

**Lemma 5.8** The equation \( \sum_{s=1}^{k} \cos(sx) = -1/2 \) has \( 2k \) solutions in the interval \([-\pi, \pi]\), namely \( \pm 2j\pi/(2k+1), j \in \{1, \ldots, k\} \).
Proof. Observe that the above equation has at most $2k$ solutions in the interval $[-\pi, \pi]$, as the highest multiple of $x$ is $k$ and all the multiples are integers. On the other hand we have

$$
\sum_{s=1}^{k} \cos(sx) = \sum_{s=1}^{k} \frac{e^{i\pi s} + e^{-i\pi s}}{2} = \frac{1}{2} \left( e^{i\pi} e^{i\pi s} - 1 + e^{-i\pi} e^{-i\pi s} - 1 \right) = \frac{e^{i\pi} e^{i\pi x} - 1}{2} e^{ix/2} - e^{-ix/2} = \frac{\sin(kx/2) \cos((k+1)x/2)}{\sin(x/2)} = \sin((k+1)x/2) - 1/2.
$$

Now substituting $x$ with any of the numbers $2j\pi/(2k+1)$, $j \in \{\pm 1, \ldots, \pm k\}$ we obtain

$$
\sin((2k+1)x/2) = \sin(j\pi) = 0,
$$

and thus $\sum_{s=1}^{k} \cos(sx) = -1/2$ for every $\pm 2j\pi/(2k+1), j \in \{1, \ldots, k\}$. □

**Lemma 5.9** The equation

$$
\sum_{s=1}^{k-1} \cos(sx) + \cos((k+1)x) = -1/2
$$

has $2k + 2$ solutions in the interval $[-\pi, \pi]$, namely $\pm 2j\pi/(2k+1), j \in \{1, \ldots, k\}$, and $\pm \pi/3$.

Proof. Observe that the above equation has at most $2k + 2$ solutions in the interval $[-\pi, \pi]$, as the highest multiple of $x$ is $k + 1$ and all the multiples are integers. On the other hand we have

$$
\cos\left(\frac{2s(k + 1)\pi}{2k + 1}\right) - \cos\left(\frac{2sk\pi}{2k + 1}\right) = -2 \sin(s\pi) \sin\frac{s\pi}{2k + 1} = 0,
$$

and thus $\sum_{s=1}^{k-1} \cos(sx) + \cos((k+1)x) = -1/2$ for every $\pm 2j\pi/(2k+1), j \in \{1, \ldots, k\}$ by the Lemma 5.8. We have also

$$
\sum_{s=1}^{k-1} \cos\left(\frac{s\pi/3}{2}\right) + \cos\left((k+1)\pi/3\right) = \frac{1}{2} \sum_{s=1}^{k-1} \frac{e^{i\pi s/6} - 1}{\sin(\pi/6)} - 1/2 - \cos(k\pi/3) + \cos((k+1)\pi/3) = -1/2,
$$

what finishes the proof. □

The following proposition gives the necessary conditions for $Ci(n, \{1, \ldots, k-1, k + 1\})$ to be closed distance magic.
Proposition 5.10 For given \( n \) and \( k, 1 < k \leq \lfloor (n - 3)/2 \rfloor \), let the multiplicity of \(-1\) in \( Sp(Ci(n, \{1, \ldots, k - 1, k + 1\}) \) be \( m \). Then \( m = m_1 + m_2 \), where

\[
m_1 = 2|\{t | 1 \leq t \leq k, nt \equiv 0 \text{ (mod } 2k + 1)\}|
\]

and \( m_2 = 2 \) if \( n \equiv 0 \text{ (mod } 6) \) and \( m_2 = 0 \) otherwise. In particular, if the graph \( Ci(n, \{1, \ldots, k - 1, k + 1\}) \) is closed distance magic, then \( nt \equiv 0 \text{ (mod } 2k + 1) \) for some \( t \in \{1, \ldots, k\} \) or \( n \equiv 0 \text{ (mod } 6) \).

Proof. Let \( G \cong Sp(Ci(n, \{1, \ldots, k - 1, k + 1\})) \) be closed distance magic graph. As \( G \) is regular, by Corollary 2.3 we have that \(-1 \in Sp(G)\) and hence \( \sum_{j=1}^{k-1} \cos(2j\pi s/n) + \cos(2j\pi(k + 1)/n) = -1/2 \). Putting \( x = 2j\pi/n \) in Lemma 5.9 we obtain that \( 2j\pi/n = \pm 2tn/(2k + 1) \) and so \( j = \pm tn/(2k + 1) \) for some \( t \in \{1, \ldots, k\} \), or \( 2j\pi/n = \pm \pi/3 \) and so \( j = \pm n/6 \).

6 A solution to combinatorial problem forcing more closed distance magic graphs

Let us consider graphs created in the following way: take the cycle \( C_k \), exchange every vertex \( v_i \) to a complete graph of some order (we will denote such a clique by \( K[v_i] \)) and join all the vertices of this complete graph to all the vertices of \( K[v_j] \) for every \( v_j \) being a neighbor of \( v_i \) in \( C_k \). In other words: every edge \( v_i v_j \) of initial graph (cycle) becomes a complete graph \( K_{p_i + p_j} \), where \( p_i \) and \( p_j \) are the sizes of \( K[v_i] \) and \( K[v_j] \), respectively. Let \( n = |V(G)| = p_1 + \cdots + p_k \).

In the special case, where all the complete graphs are of the same order \( p \), we obtain \( G \cong C_k \boxtimes K_p \) at the end. Hence on one hand we try to generalize the Theorems 3.2 and 3.3 by more general graphs than strong products, while on the other side we specialize these two theorems as the components under discussion are cycles instead of general regular graphs.

The following observation gives necessary conditions for a graph defined as above to be closed distance magic (in some cases).

Observation 6.1 Let \( G \) be a closed distance magic graph constructed as above. If \( k \not\equiv 0 \text{ (mod } 3) \), then the sum of labels in every clique \( K[v_i] \) equals to \( \binom{n+1}{2}/k \).
Proof. For any \(v_i \in V(C)\), let us choose any vertex \(v \in K[v_i]\). Then, \(w(v) = \sum_{u \in K[v_{i-1}] \cup K[v_{i}] \cup K[v_{i+1}]} \ell(u)\). For any \(x \in K[v_{i+1}]\) we have in turn \(w(x) = \sum_{u \in K[v_{i}] \cup K[v_{i+1}] \cup K[v_{i+2}]} \ell(u)\). As \(w(v) = w(x)\), we have \(\sum_{u \in K[v_{i}]} \ell(u) = \sum_{u \in K[v_{j}]} \ell(u)\) for any \(i, j\) since \(k \not\equiv 0 \pmod{3}\). Hence the result follows. 

The idea is then to find such a labeling of vertices, that the sum of the labels is the same in every clique. This is not always the necessary condition, but if we are able to find such a labeling, then we can also label any graph in which the role of \(C_k\) is played by any regular graph. So, the problem can be reformulated as follows.

**Problem 6.2** Let \(n\) and \(p_1, \ldots, p_k\) be positive integers such that \(p_1 + \cdots + p_k = n\). When is it possible to find a partition of the set \(\{1, \ldots, n\}\) into \(k\) sets \(A_1, \ldots, A_k\) such that \(|A_i| = p_i\) and \(\sum_{x \in A_i} x = \frac{n+1}{2}/k\) for every \(i \in \{1, \ldots, k\}\)?

As long as we know, this problem has not been studied so far. It provides a combinatoric problem, where the solution yields a closed distance magic labeling of a family of graphs as described above. To illustrate it let \(p_1 = 3, p_2 = p_3 = 4\) and \(p_4 = 5\) and consequently \(n = 16\). Sets \(A_1 = \{3, 15, 16\}, A_2 = \{1, 6, 13, 14\}, A_3 = \{2, 9, 11, 12\}, A_4 = \{4, 5, 7, 8, 10\}\) provide the desired partition in this case. Observe that \(C_4\) can be replaced by any other regular graph on 4 vertices, namely \(K_4\) or \(2K_2\). On the other hand, if \(p_1 = 2, p_2 = p_3 = 4\) and \(p_4 = 6\), then \(\sum_{x \in A_i} x\) should be also equal to 34, but \(\sum_{x \in A_1} x \leq 31\) for each partition. Also if \(k\) is even and \(p\) odd where \(p = p_1 = \cdots = p_k\), we now that there is no solution. Namely the construction yields \(3p - 1\)-regular graph \(C_k \boxtimes K_p\), which is not closed distance magic by the Observation 1.2 and hence there can not exists a partition from the Problem 6.2.

We end with two simple necessary conditions for Problem 6.2.

**Observation 6.3** If the mentioned partition exists, then \(n \equiv x \pmod{2k}\) where \(x \in \{0, -1\}\).

**Proof.** It follows from the fact that \(\frac{n+1}{2}/k\) must be an integer. 

The next condition is as follows. There always must be enough large labels so that their sum can be at least as big as the desired sum of the labels in cliques.
Observation 6.4 Assume that $p_1, \ldots, p_k$ are given in the non-decreasing order. Let $P_j = \sum_{i=1}^{j} p_i$. If the mentioned partition exists, then for any $1 \leq j \leq k$,

$$\sum_{i=n-P_j+1}^{n} i \geq j \frac{(n+1)}{2}/k.$$ 

Proof. The left side is the sum of $P_j$ largest labels, while the right side is the desired sum of the labels in $j$ smallest cliques.

References


