Graphs that are simultaneously efficient open domination and efficient closed domination graphs

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Abstract

A graph is an efficient open (resp. closed) domination graph if there exists a subset of vertices whose open (resp. closed) neighborhoods partition its vertex set. Graphs that are efficient open as well as efficient closed (shortly EOCD graphs) are investigated. The structure of EOCD graphs with respect to their efficient open and efficient closed dominating sets is explained. It is shown that the decision problem regarding whether a graph is an EOCD graph is an NP-complete problem. A recursive description that constructs all EOCD trees is given and EOCD graphs are characterized among the Sierpiński graphs.

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1 Introduction

The domination number, $\gamma(G)$, of a graph $G$ is an important classical graph invariant with many applications. It is defined as the minimum cardinality of a subset of vertices $S$, called dominating set, with the property that each vertex from $V(G) - S$ has a neighbor in $S$. A dominating set $S$ of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. The union of closed neighborhoods centered in vertices of a dominating set covers the entire vertex set. A classical question for a cover of a set is: when does this cover form a partition? A graph $G$ is called an efficient closed domination graph, or ECD graph for short, if there exists a set $P$, $P \subseteq V(G)$, such that the closed neighborhoods centered in vertices
of $P$ partition $V(G)$. Such a set $P$ is called a perfect code of $G$. More general, a set $P$ is an $r$-perfect code of $G$ if the $r$-balls centered in vertices of $P$ partition $V(G)$.

The study of perfect codes in graphs was initiated by Biggs [5] and presents a generalization of the problem of the existence of (classical) error-correcting codes. The initial research focused on distance regular and related classes of graphs, while later the investigation was extended to general graphs, cf. [33]. To determine whether a given graph has a 1-perfect code is an NP-complete problem [3] and remains NP-complete on $k$-regular graphs ($k \geq 4$) [34], on planar graphs of maximum degree 3 [13, 34], as well as on bipartite and chordal graphs [40]. On the positive side, the existence of a 1-perfect code can be decided in polynomial time on trees [13], interval graphs [35], and circular-arc graphs [29].

In the last period, the study of perfect codes in graphs was primarily focused on their existence and construction in some central families of graphs. Many researches were done on standard graph products and product-like graphs [2, 23, 28, 39, 42, 44]. Among other classes of graphs on which perfect codes were investigated we mention Sierpiński graphs [8, 27], cubic vertex-transitive graphs [31], circulant graphs [10], twisted tori [24], dual cubes [25], and AT-free and dually chordal graphs [4].

A graph invariant closely related to the domination number is the total domination number $\gamma_t(G)$ [21]. It is defined as the minimum cardinality of a subset of vertices $D$, called total dominating set, such that each vertex from $V(G)$ has a neighbor in $D$. A total dominating set $D$ of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. If we switch to neighborhoods, the union of open neighborhoods centered in vertices of a total dominating set covers the entire vertex set and again one can pose the question: when does this cover form a partition? A graph $G$ is called an efficient open domination graph, or an EOD graph for short, if there exists a set $D$, $D \subseteq V(G)$, such that open neighborhoods centered in vertices of $D$ partition $V(G)$. Such a set $D$ is called an EOD set. Note that two different vertices of an EOD set are either adjacent or at distance at least 3.

The problem of deciding whether a graph $G$ is an EOD graph is NP-complete [16, 38]. For various properties of EOD graphs see [15], a recursive characterization of EOD trees is given in [16]. EOD graphs that are also Cayley graphs were studied in [41], while EOD grid graphs were investigated in [7, 9, 30]. EOD direct product graphs were characterized in [1], for other standard graph products (lexicographic, strong, disjunctive and Cartesian) see [36]. Domination-type problems studied on graph products are usually most difficult on the Cartesian product, recall the famous Vizing’s conjecture [6]. It is hence not surprising that EOD graphs studied on product graphs seems to be the most difficult on the Cartesian product. For some very recent results in this direction see [32].

In this paper we study the graphs that are ECD and EOD at the same time and call them efficient open closed domination graphs, EOCD graphs for short. In the rest of the paper we shall use the term ECD set instead of 1-perfect code to make the notation consistent.

We proceed as follows. In the rest of this section additional definitions are given and a basic result recalled. In the next section we show how to construct an ECD graph
from and EOD graph and vice versa, and consider the structure of EOCD graphs from
the viewpoint of the relationship between selected EOD sets and selected ECD sets.
In two extremal cases we find that for the corresponding EOCD graphs \( G \) we have
\( \gamma_t(G) = \gamma(G) \) and \( \gamma_t(G) = 2\gamma(G) \), respectively. In Section 3 we prove that the decision
problem regarding whether a graph is an EOCD graph is an NP-complete problem. On
the other hand, in one of the above extremal cases, EOCD graphs can be recognized
in polynomial time. Then, in Section 4 we give a recursive description of EOCD trees,
while in the final section EOCD graphs are characterized among the Sierpiński graphs.

We will use the notation \([n] = \{1, \ldots, n\}\) and \([n]_0 = \{0, \ldots, n - 1\}\). Throughout
the article we consider only finite, simple graphs. If \( S \) is a subset of vertices of a graph,
then \( \langle S \rangle \) denotes the subgraph induced by \( S \). A matching of a graph is an independent
set of its edges. For the later use we next state the following basic result. Its first
assertion has been independently discovered several times, cf [19, Theorem 4.2], while
for the second fact see [36].

**Proposition 1.1** Let \( G \) be a graph.

(i) If \( P \) is an ECD set of \( G \), then \( |P| = \gamma(G) \).

(ii) If \( D \) is an EOD set of \( G \), then \( |D| = \gamma_t(G) \).

2 **On the structure of EOCD graphs**

In this section we first show that each EOD graph naturally yields an ECD graph
and that each ECD graph can be modified to an EOD graph. Then we consider the
structure of EOCD graphs with respect to the relationship between their EOD in ECD
sets.

If \( D \) is an EOD set of a graph \( G \), then \( D \) induces a matching \( M \). Note that an edge
from \( M \) lies in no triangle, hence its contracting produces no parallel edges. Now, let
\( G' \) be the graph obtained from \( G \) by contraction all the edges from \( M \). Then \( G' \) is an
ECD graph with an ECD set consisting of the vertices obtained by the contraction of
\( M \).

Conversely, let \( G' \) be an ECD graph with an ECD set \( P \). For every vertex \( v \in P \)
partition the set of its neighbors arbitrarily into sets \( A \) and \( B \). (If the degree of \( v \) is
1, then necessarily one of these sets is empty.) Let \( G \) be the graph obtained from \( G' \)
by replacing every vertex \( v \in P \) by two adjacent vertices \( v_A \) and \( v_B \), and adding edges
\( uv_A \) for every \( u \in A \) and edges \( uv_B \) for every \( u \in B \). Then \( G \) is an EOD graph with an
EOD set \( \{v_A, v_B : v \in P\} \).

Let \( G \) be an EOCD graph with an EOD set \( D \) and an ECD set \( P \). Then \( V(G) \) can
be partitioned into sets \( D \cap P, D - P, P - D, \) and \( R = V(G) - (D \cup P) \), see Fig. 1.
Clearly, some of these sets may be empty. From the definitions of ECD and EOD sets
we infer the following properties.

- A vertex from \( D \cap P \) (a black squared vertex in Fig. 1) can have an arbitrary
  number of neighbors in \( R \), has a unique neighbor in \( D - P \), and has no neighbors
  in \( P - D \).
• A vertex from \( P - D \) (a white squared vertex in Fig. 1) can have an arbitrary number of neighbors in \( R \), a unique neighbor in \( D - P \), and no neighbors in \( D \cap P \).

• A vertex from \( D - P \) (a black vertex in Fig. 1) can have an arbitrary number of neighbors in \( R \) and, either a unique neighbor in \( P - D \) and a unique neighbor in \( D - P \), or a unique neighbor in \( D \cap P \).

• A vertex from \( R \) (a white vertex in Fig. 1) can have an arbitrary number of neighbors in \( R \) and either a unique neighbor in \( P - D \) and a unique neighbor in \( D - P \), or a unique neighbor in \( D \cap P \).

• Vertices from \( D \cap P \) together with their unique neighbors from \( D - P \) induce a matching.

• Vertices from \( P - D \) together with their unique neighbors in \( D - P \) induce \( k \) copies of \( P_4 \), where \( 2k = |P - D| \).

The described structure is visible in Fig. 1. The same notation will be used later in Fig. 3.

![Diagram of an EOCD graph](image)

**Figure 1:** Structure of an EOCD graph.

The described structure above yields two extreme cases: either \( D \cap P = \emptyset \) or \( P - D = \emptyset \). Clearly, the structure of any EOCD graph depends on the chosen ECD set \( P \) and EOD set \( D \). That is, different pairs of sets \( P, D \) could produce different configurations. In this sense, if there exists an ECD set \( P \) and an EOD set \( D \) in \( G \), such that \( D \cap P = \emptyset \), then we say that \( G \) is an EOCD graph with empty \( D \cap P \), and if \( P - D = \emptyset \), then we say that \( G \) is an EOCD graph with empty \( P - D \). We observe that \( D - P \) is always non-empty for every ECD set \( P \) and every EOD set \( D \) of any EOCD graph. Moreover, if \( R = \emptyset \), then \( G \) is formed only from the disjoint union of copies of \( K_2 \) and copies of \( P_4 \).

The following two propositions follow directly from the above mentioned structure. The first result characterizes the EOCD graphs with empty \( D \cap P \).

**Proposition 2.1** A graph \( G \) is an EOCD graph with empty \( D \cap P \) if and only if there exists \( A \subseteq V(G) \), such that \( \langle A \rangle = kP_4 \), where every vertex from \( V(G) - A \) is adjacent to exactly one vertex of degree 1 in \( \langle A \rangle \) and one vertex of degree 2 in \( \langle A \rangle \).
The second result characterizes the EOCD graphs with empty $P - D$.

**Proposition 2.2** A graph $G$ is an EOCD graph with empty $P - D$ if and only if there exists $D \subseteq V(G)$ that induces a matching $M$, where every edge of $M$ contains at least one vertex of degree 1 in $G$ (this vertex is from $D - P$) and every vertex from $V(G) - D$ is adjacent to exactly one vertex in $M$ which is in $P$.

We end this section with a connection between $\gamma(G)$ and $\gamma_t(G)$ for EOCD graphs with empty $D \cap P$ or empty $P - D$, respectively. Both results follow from the described structure of EOCD graphs, and by applying Proposition 1.1.

**Proposition 2.3** If $G$ is an EOCD graph with empty $D \cap P$, then $\gamma_t(G) = \gamma(G)$.

**Proposition 2.4** If $G$ is an EOCD graph with empty $P - D$, then $\gamma_t(G) = 2 \gamma(G)$.

Recall that for any graph $G$ (without isolated vertices) $\gamma(G) \leq \gamma_t(G) \leq 2 \gamma(G)$ holds. The above two propositions are of interest particularly because it is an open problem to characterize the graphs $G$ with $\gamma_t(G) = 2 \gamma(G)$, as well as the graphs $G$ with $\gamma_t(G) = \gamma(G)$, cf. [21] p. 36. In this direction, the trees $T$ for which $\gamma_t(T) = \gamma(T)$ holds were characterized in [12] Theorem 6 as the trees obtained from a disjoint union of $P_3$s by means of certain four operations. Moreover, a characterization of trees $T$ for which $\gamma_t(T) = 2 \gamma(T)$ holds was obtained in [20]. For these two results see also [21, Sections 4.6 and 4.7].

3 Complexity results

In this section we deal with the problem of deciding whether a given graph contains an EOD set and an ECD set (EOCD Problem for short), that is, the following problem.

**EOCD Problem**

*Input:* A simple graph $G$.

*Question:* Is $G$ an EOCD graph?

In order to study this problem, we shall make a reduction from the one-in-three 3-SAT problem, which is known to be NP-complete [14] and reads as follows.

**One-In-Three 3-SAT**

*Input:* A Boolean formula $F$ on $n$ variables and $m$ clauses.

*Question:* Is there a satisfying truth assignment for the $n$ variables, such that each clause has exactly one true literal?

Next we present the main result of this section, which is in part inspired by the proof of the NP-completeness of the problem of deciding whether a graph contains an EOD set given in [16].
Theorem 3.1 The EOCD Problem is NP-complete.

Proof. It is clear that the EOCD Problem is in NP, since verifying that a given set of vertices of a graph \( G \) is an EOD set or an ECD set can be done in polynomial time. We consider now a Boolean formula \( \mathcal{F} \) with variables \( X = \{x_1, \ldots, x_n\} \) and clauses \( C = \{c_1, \ldots, c_m\} \). Each clause contains three literals, each of which we shall denote by \( x_i \) for a positive literal, or by \( \overline{x}_i \) for a negative one. From the formula \( \mathcal{F} \), we construct a graph \( G_\mathcal{F} \) in the following way. For any variable \( x_i \in X \), add to \( G_\mathcal{F} \) the graph \( G_i \) from Fig. 2. For each clause \( c_i \in C \), we add a vertex \( y_i \). Now, if a variable \( x_i \) occurs as a positive literal in a clause \( c_j \), then add the edge \( y_j u_i \), otherwise (if a variable \( x_i \) occurs as a negative literal in a clause \( c_j \)) add the edge \( y_j \overline{u}_i \). Clearly, \( G_\mathcal{F} \) can be constructed in polynomial time.

![Figure 2: The graph \( G_i \) corresponding to a variable \( x_i \).](image)

We claim that \( G_\mathcal{F} \) is an EOCD graph if and only if there is a satisfying truth assignment for the \( n \) variables in the Boolean formula \( \mathcal{F} \), such that each clause has exactly one true literal, that is, if and only if \( \mathcal{F} \) has a one-in-three satisfying truth assignment.

Assume first that \( \mathcal{F} \) has a one-in-three satisfying truth assignment. We construct two sets \( D \) and \( P \) in the following way. Add to \( D \) the vertices \( q_i, c_{i1}, c_{i4}, c_{i5} \), and to \( P \) the vertices \( q_i, c_{i3}, c_{i6} \) for every \( i \in [n] \). Now, if the variable \( x_i \) is assigned the value true, then we add to \( D \) the vertices \( u_i, v_{i1}, w_{i4}, w_{i5} \), and to \( P \) the vertices \( u_i, w_{i3}, w_{i7} \). On the other hand, if \( x_i \) is assigned the value false, then we add to \( D \) the vertices \( \overline{u}_i, v_{i2}, w_{i3}, w_{i4} \), and to \( P \) the vertices \( \overline{u}_i, w_{i2}, w_{i5} \). It is easy to see that \( D \cap V(G_i) \) is an EOD set and \( P \cap V(G_i) \) an ECD set of \( G_i \). Moreover, since the truth assignment has exactly one literal with value true, each vertex \( y_j \), with \( j \in [m] \), is adjacent to exactly 6
one vertex of $D$ and exactly one vertex of $P$ (clearly both vertices coincide). Thus, $D$ is an EOD set and $P$ is an ECD set in $G_F$ and, as a consequence, $G_F$ is an EOCD graph.

Conversely, assume that $G_F$ is an EOCD graph. Let $D$ be an EOD set and $P$ an ECD set in $G_F$. We next collect several facts regarding the sets $D$ and $P$.

- The vertex $q_i$ ($i \in [n]$) lies in $D \cap P$. Indeed, this fact follows because $q_i$ is adjacent to leaves $t_{i3}$ and $t_{i4}$.
- The vertices $c_{i1}, c_{i4}, c_{i5}$ ($i \in [n]$) belong to $D$. The vertex $c_{i1}$ belongs to $D$ because otherwise the 7-cycle on vertices $c_{ij}$ cannot be efficiently open dominated. We then consequently see that also $c_{i4}, c_{i5} \in D$.
- The vertices $t_{i2}, t_{i3}, t_{i4}, c_{i1}$ ($i \in [n]$) do not lie in $P$, and the vertices $c_{i3}, c_{i6}$ ($i \in [n]$) lie in $P$. These facts follow immediately from the first point.
- Either $(u_i \notin D$ and $\overline{w_i} \in D$) or $(u_i \in D$ and $\overline{w_i} \notin D$). Similarly, either $(u_i \notin P$ and $\overline{w_i} \in P$) or $(u_i \in P$ and $\overline{w_i} \notin P$). Indeed, since $q_i \in D \cap P$ and $c_{i1} \in D$, for every $i \in [n]$, the vertices $t_{i1}, t_{i2} \notin D \cup P$. Thus, $t_{i1}$ must be dominated either by $u_i$ or by $\overline{w_i}$ in $D$ and in $P$.
- If $u_i \in P$ and $\overline{w_i} \notin P$, then $v_{i1} \notin P$ and every vertex $y_j$ such that the variable $x_i$ belongs to the clause $c_j$ does not belong to $P$. Moreover, to efficiently dominate the vertices $v_{i2}, w_{i1}, \ldots, w_{i7}$ we clearly have that $w_{i3} \in P$ and exactly one vertex of the pair $w_{i6}, w_{i7}$ belongs to $P$.
- Analogously, if $u_i \notin P$ and $\overline{w_i} \in P$, then we obtain that $v_{i2} \notin P$ and every vertex $y_j$ such that the variable $x_i$ belongs to the clause $c_j$ does not belong to $P$. Also, $w_{i5} \in P$ and exactly one vertex of the pair $w_{i1}, w_{i2}$ belongs to $P$.
- If $u_i \in D$ and $\overline{w_i} \notin D$, then either $v_{i1} \in D$ or there exists a vertex $y_j \in D$ such that the variable $x_i$ appears as positive in the clause $c_j$. If the latter happens ($y_j \in D$), then $v_{i1} \notin D$. It is straightforward to observe that, in such a case, any subset of vertices of the set $\{v_{i2}, w_{i1}, \ldots, w_{i7}\}$ does not efficiently open dominate the same set of vertices $\{v_{i2}, w_{i1}, \ldots, w_{i7}\}$, which is a contradiction. Thus, $y_j \notin D$ and therefore $v_{i1} \in D$. We also observe that $w_{i4}, w_{i5} \in D$.
- Analogously to the last item, if $u_i \notin D$ and $\overline{w_i} \in D$, then $v_{i2}, w_{i3}, w_{i4} \in D$.

As a consequence of the above facts, we have that either $u_i, v_{i1} \in D$ or $\overline{w_i}, v_{i2} \in D$, and either $u_i \in P$ or $\overline{w_i} \in P$. Now, we say that a subgraph $G_i$ of $G_F$, corresponding to a variable $x_i$, is nice if either $u_i \in D \cap P$ or $\overline{w_i} \in D \cap P$. Assume that there exists $G_i$ which is not nice, i.e., $u_i, \overline{w_i} \notin D \cap P$. Hence, either $u_i \in D$ and $\overline{w_i} \in P$ or $u_i \in P$ and $\overline{w_i} \in D$. Consider a clause $c_j$ such that $x_i \in c_j$. Hence, $y_j$ is dominated either by $u_i$ or by $\overline{w_i}$ from $G_i$, which means either by $D$ or by $P$, say $D$. Therefore, there must exist another variable $x_k \in c_j$, such that $y_j$ is dominated also by $P$ and not by $D$. This implies that $G_k$ is not nice as well. Notice that for the third literal $x_k \in c_j$, $G_k$ must
be nice. In general, for every clause $c_j$, either all three corresponding graphs are nice, or exactly one is nice and two are not. Moreover, if the later is true, then $y_j$ is not dominated from $D$ and from $P$ by the nice subgraph.

Let $P_i = P \cap V(G_i)$ ($i \in [n]$). For every not nice graph $G_i$ we exchange some vertices of $P_i$ as follows. If $u_i \in P_i$, then $P_i' = (P_i - \{w_i, w_i3, w_i6, w_i7\}) \cup \{w_i, w_i2, w_i5\}$, and if $\overline{u_i} \in P_i$, then $P_i' = (P_i - \{\overline{w_i}, w_i5, w_i1, w_i2\}) \cup \{u_i, w_i3, w_i7\}$. If $G_i$ is nice, then $P_i' = P_i$. We claim that $P' = \bigcup_{i=1}^m P_i'$ is an ECD set, such that together with $D$ every subgraph $G_i$ is nice. Clearly $P_i'$ is an ECD set for $G_i$ by the items above. If some $y_j$ was dominated by a vertex $u_i$ (or by $\overline{w_i}$) which was in $P_i$ but not now in $P_i'$, then $y_j$ is now dominated either by $u_i$ or by $\overline{w_i}$, where $x_i$ and $x_i'$ are those variables from the clause $c_j$, for which $G_i$ and $G_i'$ were not nice. Thus, $P'$ is an ECD set. Moreover, the EOD set $D$ and the ECD set $P'$ lead to the fact that every $G_i$ is nice. Since $D$ is an EOD set and $P'$ is an ECD set, then every vertex $y_j$ corresponding to a clause is adjacent to exactly one vertex $u_i$ or $\overline{w_i}$ of $G_i$. Now, if $u_i \in D \cap P'$, then we set the variable $x_i$ as true, otherwise (if $\overline{w_i} \in D \cap P'$) set $x_i$ as false. It clearly follows that such an assignment is a truth assignment in exactly one literal in every clause for $F$ and the proof is complete.

\[\square\]

In view of Theorem 3.1 it is reasonable to try to find some special classes of graphs for which the EOCD Problem is polynomial. Simple examples are provided by the paths $P_n$ which are EOCD graphs if and only if $n \not\equiv 1 \pmod{4}$ and the cycles $C_n$ which are EOCD graphs if and only if $n \equiv 0 \pmod{12}$. Note also that a complete bipartite graph $K_{r,t}$ is an EOCD graph if and only if $r = 1$ or $t = 1$. Moreover, the hypercube $Q_n$ is an EOCD graph if and only if $n = 1$. Indeed, suppose that $Q_n$, $n \geq 1$, is an EOCD graph. As $Q_n$ is $n$-regular, its order must be divisible by $n$ (because it admits an EOD set) as well as by $n + 1$ (since it admits an ECD set). Since the order of $Q_n$ is $2^n$, this is only possible if $n = 1$.

We end this section with a discussion on extreme cases with respect to the structure of EOCD graphs as described in Section 2.

**Theorem 3.2** If $G$ is a graph on $n$ vertices and $m$ edges, then it can be decided in $O(nm)$ time whether $G$ is an EOCD graph with empty $P - D$.

**Proof.** Let $G$ be a graph. Clearly, components which are isomorphic to $K_2$ (if they exist) do not influence the fact that $G$ is an EOCD graph or not. Hence we may restrict our attention to the case when $G$ has no components isomorphic to $K_2$. If there exists no degree 1 vertex, then by Proposition 2.2 $G$ is not an EOCD graph with empty $P - D$. Let $P$ be the set of all support vertices of degree one vertices. For every support from $P$ choose exactly one neighbor of degree 1 and let $D$ be a set containing $P$ as well as the chosen vertices of degree 1. By Proposition 2.2 one only needs to check if $D$ and $P$ are an EOD set and an ECD set of $G$, respectively. Even more, it is clear that $P$ is an ECD set in $G$ if and only if $D$ is an EOD set of $G$. Hence it is enough to check whether the union of closed neighborhoods centered in $P$ covers $V(G)$ and whether these closed neighborhoods have pairwise empty intersection. The first
task can be clearly done in $O(m)$ time. For the second task it suffices to check if the
distance between any two different vertices from $P$ is at least 3. Clearly, this can be
done in time $O(mn)$, if we start the BFS algorithm in an arbitrary vertex of $P$. □

We end the section with a question about the other extremal case.

**Problem 3.3** Can it be checked in polynomial time whether $G$ is an EOCD graph with
empty $D \cap P$?

4 EOCD trees

Let $T'$ be an EOCD tree with an EOD set $D'$ and an ECD set $P'$. We now introduce
five operations that construct larger EOCD trees from $T'$. In the main theorem of this
section we will then prove that these operations are characteristic for EOCD trees. The
operations are illustrated in Fig. 3 where we use the convention introduced in Section 2:
a vertex from $D \cap P$ is black squared, a vertex from $P - D$ is white squared, a vertex
from $D - P$ is black circled, and the remaining vertices are white circled.

(O1) For $u \in D' \cap P'$ we obtain $T$ from $T'$ by adding a vertex $v$ and edge $uv$.

(O2) For $w \notin D'$ we obtain $T$ from $T'$ by adding a path $xuv$ and edge $wx$.

(O3) For $t \in D' - P'$ we obtain $T$ from $T'$ by adding a path $zxuv$ and edge $tz$.

(O4) For a path $vux$ with $\deg(v) = 1$, $\deg(u) = 2$, $u, x \in D'$, and $u \in P'$, we obtain $T$
from $T'$ by adding a vertex $y$ and edge $yx$.

(O5) For a path $uxwzuv'$ with $\deg(u) = \deg(x') = 1$, $\deg(x) = \deg(w) = \deg(w') = 2$,
$u, x, w', x' \in D'$, and $x, w' \in P'$, we obtain $T$ from $T'$ by adding a vertex $v$ and
edge $uv$.

The main difference between these five operations is that for $O_1$ the original EOD
set and ECD set do not change, for $O_2$ and $O_3$ we add some vertices to the EOD set and
to the ECD set, while for $O_4$ and $O_5$ the EOD set remains the same and we exchange
some vertices in the ECD set.

**Theorem 4.1** A tree $T$ is an EOCD graph if and only if $T$ can be obtained from $K_2$
by a sequence of operations $O_1 - O_5$.

**Proof.** Assume first that $T$ is a tree obtained from $K_2$ by a sequence of operations
$O_1 - O_5$. We will show that $T$ is an EOCD tree by induction on the length $k$ of the
mentioned sequence. If $k = 0$, then $T \cong K_2$ which is an EOCD graph. Let now $k > 0$
and let $T'$ be a tree obtained from $K_2$ by using the same sequence as for $T$, but without
including the last step. By the induction hypothesis, $T'$ is an EOCD tree with an EOD
set $D'$ and an ECD set $P'$. If $T$ is obtained from $T'$ by operation $O_1$, then clearly $T$
is an EOCD tree for $D = D'$ and $P = P'$ (see the upper left diagram of Fig. 3). If $T$
is obtained from $T'$ by operation $O_2$, then $T$ is an EOCD tree where $D = D' \cup \{u, v\}$. The set $P$ depends whether $w$ is in $P'$ or not. If $w \in P'$, then $P = P' \cup \{v\}$, and if $w \notin P'$, then $P = P' \cup \{u\}$ (see the diagrams of the second line of Fig. 3). Suppose now that we apply operation $O_3$ on $T'$ to get $T$. Again it is straightforward to see that $T$ is an EOCD graph for $D = D' \cup \{u, x\}$ and $P = P' \cup \{v, w\}$ (see the upper right diagram of Fig. 3). If operation $O_4$ is applied to get $T$ from $T'$, then we set $D = D'$ and $P = (P' - \{u\}) \cup \{v, y\}$ and $T$ is an EOCD tree again (see the diagram in the third line of Fig. 3). Finally, if $T$ is obtained from $T'$ by operation $O_5$, then it is not hard to see that $T$ is an EOCD tree for $D = D'$ and $P = (P' - \{x, w'\}) \cup \{v, x', w\}$ (see the lower diagram of Fig. 3).

To prove the converse, let $T$ be an EOCD tree with an EOD set $D$ and an ECD set $P$. Let $r \in V(T)$ be a vertex of $T$ and consider $T$ as a rooted tree with the root $r$. Let $v$ be a vertex of degree 1 of $T$ that is at the maximum distance from $r$ and let $u$ be the support vertex of $v$. Clearly $u \in D$, while either $u \in P$ or $v \in P$. We call a neighbor $y$ of $x$ a down- (resp. up-) neighbor of $x$ if $y$ is further (resp. closer) from $r$ than $x$. We proceed by induction on the number of vertices of $T$. Clearly, $K_2$ is the smallest EOCD tree, hence the base of the induction. We distinguish the following cases.

**Case 1:** $v \notin P$ and $v \notin D$.

In this case $u \in P \cap D$. We obtain a tree $T'$ from $T$ by deleting $v$. Clearly $T'$ is an EOCD tree with $D' = D$ and $P' = P$. By the induction hypothesis $T'$ can be built...
from $K_2$ by a sequence of operations $O_1 - O_5$. If we add the operation $O_1$ at the end of this sequence, then we obtain $T$ from $K_2$ by a sequence of operations $O_1 - O_5$.

**Case 2:** $v \notin P$ and $v \in D$.  
Then $u \in P \cap D$. If $\deg(u) = 1$, then $T \cong K_2$ and we are done. So, let $\deg(u) > 1$. If $u$ is the support for more degree 1 vertices than $v$, then we have Case 1. (Notice that the same does occur when $u = r$.) Thus let $\deg(u) = 2$. Let $x$ be the up-neighbor of $u$. If $\deg(x) > 2$, then $x$ has a down-neighbor $y$ different from $u$. If $\deg(y) = 1$, then we have a contradiction with $P$ being an ECD set of $T$, since $u \in P$ implies that $y$ and $x$ cannot be in $P$ and therefore $y$ is neither dominated by $P$ nor $y \in P$. So $\deg(y) > 1$ and let $y'$ be a down-neighbor of $y$. Clearly, $\deg(y') = 1$ by the choice of $v$. This yields a contradiction with $D$ being an EOD set of $T$, since $y'$ cannot be in $D$ because $x$ is already dominated by $u \in D$. Thus, $\deg(x) = 2$ and let $w$ be the up-neighbor of $x$ or the other down-neighbor when $x = r$. By the choice of $v$, $x$ must be different from $r$ or we obtain the same problems as for $\deg(x) > 2$. Since $x$ is the neighbor of $u \in D \cap P$, we have that $w \notin D$ and $w \notin P$. Let $T'$ be the tree obtained from $T$ by deleting vertices $v,u,x$. Then $T'$ is an EOCD tree with $D' = D - \{u,v\}$ and $P' = P - \{u\}$. By the induction hypothesis, $T'$ can be built from $K_2$ by a sequence of operations $O_1 - O_5$. Adding operation $O_2$ at the end of this sequence we obtain $T$ from $K_2$ by a sequence of operations $O_1 - O_5$.

**Case 3:** $v \in P \cap D$.  
If $\deg(u) = 1$, then $T \cong K_2$ and we are done. If $\deg(u) > 2$, then we have a contradiction with $P$ being an ECD set. Thus $\deg(u) = 2$. Also notice that $u \neq r$, since otherwise $T$ would be isomorphic to $P_3$, which is not possible with $v \in P$. Let $x$ be the up-neighbor of $u$. Clearly $x \notin D \cup P$. If $x$ would have a down-neighbor different from $u$ or if $x = r$, then we have a contradiction with $D$ being an EOD set of $T$. Thus, also $\deg(x) = 2$ and let $w$ be the up-neighbor of $x$. Notice that now $w$ must be in $P$ to dominate $x$, but again $w \notin D$. Let $T'$ be the tree obtained from $T$ by deleting vertices $v,u,x$. Clearly $T'$ is an EOCD tree with $D' = D - \{u,v\}$ and $P' = P - \{v\}$. By the induction hypothesis $T'$ can be built from $K_2$ by a sequence of operations $O_1 - O_5$. Adding the corresponding operation $O_2$ at the end of this sequence we obtain $T$ from $K_2$ as desired.

**Case 4:** $v \in P$ and $v \notin D$.  
In this case $u \notin P$. If $u = r$, then we have a contradiction with $P$ being an ECD set when $\deg(u) > 1$ and with $D$ being an OED set if $\deg(u) = 1$. So we may assume that $u \neq r$. Clearly $\deg(u) = 2$, otherwise we have a contradiction again with $P$ being an ECD set of $T$ and by the choice of $v$. Let $x$ be the up-neighbor of $u$. Since $v \notin D$ and $v \in P$, we have that $x \in D$ and $x \notin P$, respectively. The only second down-neighbor of $x$ is $v$, otherwise we have a contradiction with $D$ being an EOD set for $T$ according to the choice of $v$. Suppose that $x$ has a down-neighbor $y$ of degree 1. Clearly, $v \in P$ implies $x \notin P$ and therefore $y \in P$ and $y$ is the unique down-neighbor of $x$ of degree 1. Thus $vuxy$ is a path. Let $T'$ be a tree obtained from $T$ by deleting the vertex $y$. Clearly $T'$ is an EOCD tree with $D' = D$ and $P' = (P - \{v,y\}) \cup \{u\}$. By the induction hypothesis $T'$ can be built from $K_2$ by a sequence of operations $O_1 - O_5$. If we add the operation $O_1$ at the end of this sequence, then we obtain $T$ from $K_2$ by a sequence of
operations $O_1 - O_5$.

Suppose now that $x$ has no down-neighbor of degree 1. If $x = r$, then we have a contradiction with $P$ being an ECD set for $T$. Hence $x \neq r$ and $\deg(x) = 2$ holds. Let $w$ be the up-neighbor of $x$. Clearly, $w \in P$ and $w \notin D$. If $\deg(w) \geq 3$, then $w$ has a down-neighbor $x'$ other than $x$. To dominate $x'$ from $D$, the vertex $x'$ must have a down-neighbor $u'$ which is in $D$ and the same holds for $u'$, which must have a down-neighbor $v'$ which is also in $D$. Moreover, to dominate $x'$, $u'$ and $v'$ from $P$ exactly once, also $v' \in P$ holds. Notice that $\deg(x') = 2$ according to that $D$ is an EOD set of $T$, and $\deg(u') = 2$ since $P$ is an ECD set of $T$. The situation for $v'$ is now as in Case 3 and we are done if $\deg(w) \geq 3$.

Thus, from now on, we consider $\deg(w) = 2$ and let $z$ be the up-neighbor of $w$ (or down-neighbor if $w = r$). Again $z \notin P$ since $w \in P$, and $z \notin D$ since $x \in D$. We consider the following subcases.

**Subcase 4.1:** $\deg(z) \geq 3$.

Let $u' \neq w$ be a down-neighbor of $z$. Since $z$ is dominated from $P$ by $w$, $w'$ is not in $P$ and therefore, $w'$ must have a down-neighbor $x'$ which is in $P$. Also, $x'$ cannot have a second down-neighbor by the choice of $v$ and the structure of $P$. We consider two possibilities regarding the vertex $w'$.

**Subcase 4.1.1:** $w' \notin D$.

In this subcase $w'$ must be dominated by a down-neighbor in $D$. If $x' \notin D$, then we have a contradiction since $x'$ has no second-down neighbor and $D$ is an EOD set. Hence, $x' \in D$ and $x'$ must have a down-neighbor $u' \in D$ for $x'$ to be dominated by $D$. Clearly $u' \notin P$. If $x'$ has another down-neighbor $u''$, then we obtain a tree $T'$ from $T$ by deleting $v$. Clearly $T'$ is an EOCD tree with $D' = D$ and $P' = P$. By the induction hypothesis, $T'$ can be built from $K_2$ by a sequence of operations $O_1 - O_5$. If we add the operation $O_1$ at the end of this sequence, then we obtain $T$ from $K_2$ by a sequence of operations $O_1 - O_5$. So we may assume that $\deg(x') = 2$. If $\deg(u') = 2$, then let $T'$ be a tree obtained from $T$ by deleting vertices $u', x', w'$. Clearly $T'$ is an EOCD tree with $D' = D - \{u', x'\}$ and $P' = P - \{x'\}$. By the induction hypothesis, $T'$ can be built from $K_2$ by a sequence of operations $O_1 - O_5$. Adding the corresponding operation $O_2$ at the end of this sequence we obtain $T$ from $K_2$ as desired. On the other hand, if $w'$ has a down-neighbor $x''$ other than $x'$, then $x''$ is not in $D$ and not in $P$, since $w'$ is already dominated by $x'$ in both $P$ and $D$. Moreover, $x''$ must be dominated by its down-neighbor $u''$ in both $P$ and $D$. Furthermore, $u''$ is dominated by its down-neighbor $v''$ in $D$. If $\deg(u'') > 2$, then we have Case 1. So let $\deg(u'') = 2$. If $\deg(x'') > 2$, we have a contradiction with $D$ being an EOD set of $T$ or by the choice of $v$. Hence, $\deg(x'') = 2$ and we proceed like in Case 2 for $v'', u'', x''$.

**Subcase 4.1.2:** $w' \in D$.

Since $z \notin D$, also in this subcase $w'$ must have a down-neighbor in $D$. Suppose first that $x' \in D$ (and recall that $x' \in P$). If $x'$ has a down-neighbor $u'$, then $\deg(u') = 1$ by the choice of $v$ and since $P$ is an ECD set. Let $T'$ be a tree obtained from $T$ by deleting $u'$. Clearly $T'$ is an EOCD tree with $D' = D$ and $P' = P$. By the induction hypothesis $T'$ can be built from $K_2$ by a sequence of operations $O_1 - O_5$, and attaching
operation $O_1$ to this sequence we obtain $T$ from $K_2$ as desired. Thus, we may assume that $\deg(v') = 1$. Observe that $\deg(u') = 2$, otherwise we have a contradiction with the choice of $v$, since $P$ is an ECD set and since $D$ is an EOD set. Let $T'$ be a tree obtained from $T$ by deleting $v$. Clearly $T'$ is an EOCD tree with $D' = D$ and $P' = (P - \{x', w, v\}) \cup \{x, w\}$. By the induction hypothesis, $T'$ can be built from $K_2$ by a sequence of operations $O_1 - O_5$. Now, adding operation $O_5$ at the end of such sequence produces our desired result. Next, let $x' \notin D$. Clearly $\delta(x') = 1$, since $D$ is an EOD set and by the choice of $v$. Let $w$ now be dominated by $x'' \in D$. To dominate $x''$ from $P$, let $w''$ be its down-neighbor. Also, $\delta(w'') = 1$ since $D$ is an EOD set and by the choice of $v$. Observe that $\delta(x'') = 2$ since any other down-neighbor $u'''$ of $x''$ would required a down-neighbor $v'$ in $P$, which is not possible since $D$ is an EOD set and by the choice of $v$. Let $T'$ be a tree obtained from $T$ by deleting vertex $x'$. Clearly $T'$ is an EOCD tree with $D' = D$ and $P' = (P - \{x', w''\}) \cup \{x''\}$. By the induction hypothesis, $T'$ can be built from $K_2$ by a sequence of operations $O_1 - O_5$. If we add operation $O_4$ at the end of this sequence, then we obtain $T$ from $K_2$ by a sequence of operations $O_1 - O_5$.

**Subcase 4.2:** $\deg(z) = 2$.

Let $T'$ be a tree obtained from $T$ by deleting $v, u, x, w, z$. Clearly $T'$ is an EOCD tree with $D' = D - \{u, x\}$ and $P' = P - \{v, w\}$. Applying the induction hypothesis once more and ending with an additional operation $O_3$, we again obtain $T$ from $K_2$ as desired and we are done. \hfill \Box

It is not obvious that all the five operations are necessary to characterize EOCD trees. To see that this is the case, note first that $P_3$ can be obtained from $K_2$ only by operation $O_1$ and that the sequence of operations $O_1, O_4$ is unique for $P_4$. Similarly, the sequence of operations $O_1, O_2$ is unique for $P_5$. To infer that operations $O_3$ and $O_5$ are also indispensable, consider the following more elaborate examples.

Let $T$ be the tree obtained from $K_{1,3}$ by subdividing one of its edges with five vertices and each of the other two edges with eight vertices. A short analysis reveals that $T$ is an EOCD tree where the vertex of degree 3 must be in $D \cap P$ and that its neighbor on the shortest leg must be in $D$. After this observation, operation $O_3$ cannot be avoided when constructing $T$ in view of Theorem 4.1. For operation $O_5$, let $P_{22}^+$ be the graph obtained from the path on 22 vertices $v_1, \ldots, v_{22}$, by adding vertices $u, w, x, y$ and edges $v_5u, uw, v_1x, xy$. One can observe that $P_{22}^+$ is an EOCD tree with a unique EOD set $D$ and a unique ECD set $P$. From here it can be concluded that operation $O_5$ is needed to build $P_{22}^+$ from $K_2$ in view of Theorem 4.1. We leave the details to the reader.

### 5 EOCD Sierpiński graphs

The Sierpiński graphs $S_p^n$ were introduced in [26] and afterwards investigated from many different aspects. Here we only mention recent studies of Sierpiński graphs related to codes and domination [11, 17, 37], their shortest paths [22, 43], and an appealing
generalization of Sierpiński graphs due to Hasumuma [18] that in turn extends several known results about Sierpiński graphs. For the additional vast bibliography on these graphs we refer to [18].

The Sierpiński graphs $S^n_p$, $p \geq 1$, $n \geq 0$, are defined as follows. $S^0_p = K_1$ for any $p$. For $n \geq 1$, the vertex set of $S^n_p$ is $[p]^n_0$, we shall denote its elements by $s = s_n \ldots s_1$. Vertices $s_n \ldots s_1$ and $t_n \ldots t_1$ are adjacent if and only if there exists a $\delta \in [n]$ such that

(i) $s_d = t_d$, for $d \in [n] - [\delta]$;

(ii) $s_\delta \neq t_\delta$; and

(iii) $s_d = t_\delta$ and $t_d = s_\delta$ for $d \in [\delta - 1]$.

Note that $S^n_1 \cong K_1$ ($n \geq 1$), $S^n_2 \cong P_2^n$ ($n \geq 1$), and $S^n_p \cong K_p$ ($p \geq 1$). Hence, for our purposes we may restrict the attention to the Sierpiński graphs $S^n_p$ with $p \geq 3$ and $n \geq 2$.

The edge set of $S^n_p$ can be equivalently defined recursively as

$$E(S^n_p) = \{is, it : i \in [p]_0, \{s, t\} \in E(S^{n-1}_p)\} \cup \{ij^{n-1}, ji^{n-1} : i, j \in [p]_0, i \neq j\}.$$ 

This implies that $S^n_p$ can be constructed from $p$ copies of $S^{n-1}_p$ as follows. For each $j \in [p]_0$ concatenate $j$ to the left of the vertices in a copy of $S^{n-1}_p$ and denote the obtained graph with $jS^{n-1}_p$. Then for each $i \neq j$ join copies $iS^{n-1}_p$ and $jS^{n-1}_p$ by the single edge $e^{(n)}_{ij} = \{ij^{n-1}, ji^{n-1}\}$.

If $1 \leq d < n$ and $s \in [p]^d_0$, then the subgraph of $S^n_p$ induced by the vertices whose labels begin with $s$ is isomorphic to $S^{n-d}_p$. It is denoted with $sS^{n-d}_p$ in accordance with the above notation $jS^{n-1}_p$. Note that $S^n_p$ contains $p^d$ pairwise disjoint subgraphs $sS^{n-d}_p$, $s \in [p]^d_0$. In particular, $S^n_p$ contains $p^{n-1}$ pairwise disjoint $p$-cliques $S^{n-1}_p$, $s \in [p]^n_0$. The vertices $i^n$, $i \in [p]_0$, of $S^n_p$ are called extreme vertices (of $S^n_p$). The clique in which an extreme vertex lies is called an extreme clique.

After this preparation we can state the following result which asserts, roughly speaking, that precisely one half of the Sierpiński graphs are EOCD graphs.

**Theorem 5.1** Let $p \geq 3$ and $n \geq 2$. Then $S^n_p$ is an EOCD graph if and only if $p$ is even.

**Proof.** Suppose that $p$ is odd and that $D$ is an EOD set of $S^n_p$. Observe first that no extreme vertex of $S^n_p$ lies in $D$ because otherwise $D$ would contain two vertices from the same extreme clique, which is not possible. Hence every vertex from $D$ is of degree $p$ and consequently $|D| = |V(S^n_p)|/p = p^{n-1}$. Since this is at the same time the number of all $p$-cliques of $S^n_p$, it follows that $D$ must have precisely one vertex in each $p$-clique of $S^n_p$. By the same argument as above, a vertex $s$ of $D$ can only be covered by a vertex $t$ of $D$ that lies in a $p$-clique that is neighboring the $p$-clique of $s$. This means that the vertices of $D$ can be partitioned into disjoint pairs. But $p$ is odd and hence $|D| = p^{n-1}$ is odd as well, hence $D$ does not exist.
Assume now that \( p \) is even, say \( p = 2k, k \geq 2 \). We first recall from \cite{27} that \( S_p^n \) contains an ECD set. In order to prove that \( S_p^n \) is an EOCD graph it thus remains to prove that it contains an EOD set. For this sake set

\[
D_{2i} = \{ s(2i)(2i+1) : s \in [p]_0^{n-2} \}, \quad 0 \leq i \leq k - 1,
\]
and

\[
D_{2i+1} = \{ s(2i+1)(2i) : s \in [p]_0^{n-2} \}, \quad 0 \leq i \leq k - 1.
\]

We claim that

\[
D = \bigcup_{i=0}^{2k-1} D_i
\]

is an EOD set of \( S_p^n \). Note first that for any \( i \in [k]_0, |D_{2i}| = |D_{2i+1}| = p^{n-2}. \) Since the sets \( D_i, i \in [2k]_0, \) are clearly pairwise disjoint, it follows that \( |D| = 2kp^{n-2} = p^{n-1}. \)

Let now \( sS_p^n, s = s_1 \ldots s_2 \in [p]_0^{n-1}, \) be an arbitrary \( p \)-clique of \( S_p^n \). If \( s_2 \) is even, say \( s_2 = 2i \), then \( s(2i+1) \in D \cap sS_p^n \), and if \( s_2 \) is odd, say \( s_2 = 2i + 1 \), then \( s(2i) \in D \cap sS_p^n \).

If follows that any \( p \)-clique contains a vertex of \( D \) and consequently it contains exactly one such vertex. Since by the construction of the sets \( D_{2i} \) and \( D_{2i+1} \) any vertex of \( D \) has a neighbor in \( D \), we conclude that \( D \) is indeed an EOD set of \( S_p^n \). \( \square \)

Combining the construction of the EOC sets in the proof of Theorem 5.1 with Proposition 1.1(ii) we get:

**Corollary 5.2** If \( p \geq 4 \) is even and \( n \geq 2 \), then \( \gamma_t(S_p^n) = p^{n-1}. \)

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