A note on the packing chromatic number of lexicographic products

Dragana Božović(1) and Iztok Peterin(1,2)

(1) Faculty of Electrical Engineering and Computer Science
University of Maribor, Koroška cesta 46, 2000 Maribor, Slovenia.
(2) Institute of Mathematics, Physics and Mechanics
Jadranska ulica 19, 1000 Ljubljana, Slovenia.
e-mails: dragana.bozovic@um.si and iztok.peterin@um.si

Abstract

The packing chromatic number \(\chi_\rho(G)\) of a graph \(G\) is the smallest integer \(k\) such that there exists a \(k\)-vertex coloring of \(G\) in which any two vertices receiving color \(i\) are at distance at least \(i + 1\). In this short note we present upper and lower bound for the packing chromatic number of the lexicographic product \(G \circ H\) of graphs \(G\) and \(H\). Both bounds coincide in many cases. In particular this happens if \(|V(H)| - \alpha(H) \geq \text{diam}(G) - 1\), where \(\alpha(G)\) denotes the independence number of \(G\).

Keywords: packing chromatic number, lexicographic product of graphs
AMS Subject Classification (2010): 05C15, 05C12, 05C70, 05C76

1 Introduction and preliminaries

Let \(G\) be a simple graph. To shorten the notation we use \(|G|\) instead of \(|V(G)|\) for the order of \(G\). The distance \(d_G(u, v)\) between vertices \(u\) and \(v\) of \(G\) is the length of a shortest path between \(u\) and \(v\) in \(G\). The diameter of \(G\) is denoted by \(\text{diam}(G)\) and is the maximum length of a shortest path between any two vertices of \(G\).

Let \(t\) be a positive integer. A set \(X \subseteq V(G)\) is a \(t\)-packing if any two different vertices from \(X\) are at distance more than \(t\). The \(t\)-packing number of \(G\), denoted by \(\rho_t(G)\), is the maximum cardinality of a \(t\)-packing of \(G\). Notice, that if \(t = 1\), then the 1-packing number equals to the independence number \(\alpha(G)\) and we use the later more common notation for it. An independent set of cardinality \(\alpha(G)\) is called \(\alpha(G)\)-set. The packing chromatic number \(\chi_\rho(G)\) of \(G\) is the smallest integer \(k\) such that \(V(G)\) can be partitioned into subsets \(X_1, \ldots, X_k\), where \(X_i\) induces an \(i\)-packing for every \(1 \leq i \leq k\). Another approach is from a \(k\)-packing coloring of \(G\), which is a function \(c : V(G) \to [k]\), where \([k] = \{1, \ldots, k\}\), such that if \(c(u) = c(v) = i\), then \(d_G(u, v) > i\). Clearly, \(\chi_\rho(G)\) is the minimum integer \(k\) for which a \(k\)-packing coloring of \(G\) exists.

\*The second author was partially supported by Slovenian research agency under the grants P1-0297 and J1-9109.
The concept of packing chromatic number was introduced by Goddard et al. in \[6\] under the name broadcast chromatic number. The problem of determining the packing chromatic number of a graph is a very difficult problem and is NP-complete even for trees as shown in \[2\]. The attention was fast drawn to Cartesian product and infinite lattices like hexagonal, triangular and similar. In \[1\] it was shown that the packing chromatic number of an infinite hexagonal lattice lies between 6 and 8. Upper bound was later improved to 7 in \[3\] and finally settled to 7 in \[9\]. For infinite triangular lattice and three-dimensional integer lattice \(\mathbb{Z}^3\) the packing chromatic number is infinite as shown in \[4\]. The packing chromatic number of the Cartesian product was already considered in \[8\] where the general upper and lower bound were set. The lower bound was later improved in \[3\]. Several exact values and bounds for special families of Cartesian product graphs can be found in \[8, 9\].

In this note we switch from Cartesian to lexicographic product and prove an upper and a lower bound for the packing chromatic number of lexicographic product. It turns out that these two bounds coincide in many cases. In particular, if \(\text{diam}(G) \leq 2\), if \(\text{diam}(G) = 3\) and \(H \not\cong K_n\) and if \(|V(H)| - \alpha(H) \geq \text{diam}(G) - 1\) and \(H \not\cong K_n\).

The lexicographic product of graphs \(G\) and \(H\) is the graph \(G \circ H\) (also sometimes denoted with \(G[H]\)) with the vertex set \(V(G) \times V(H)\). Two vertices \((g_1, h_1)\) and \((g_2, h_2)\) are adjacent if either \(g_1g_2 \in E(G)\) or \(g_1 = g_2\) and \(h_1h_2 \in E(H)\). Set \(G^h = \{(g, h) : g \in V(G)\}\) is called a \(G\)-layer through \(h\) and \(H^g = \{(g, h) : h \in V(H)\}\) is called an \(H\)-layer through \(h\). Clearly, subgraphs of \(G \circ H\) induced by \(G^h\) and \(H^g\) are isomorphic to \(G\) and \(H\), respectively. The distance between two vertices in lexicographic product is given by

\[
d_G((g_1, h_1), (g_2, h_2)) = \begin{cases} 
d_G(g_1, g_2) & : g_1 \neq g_2 \\
\min\{2, d_H(h_1, h_2)\} & : g_1 = g_2 \end{cases}
\]

and depends heavily on the distance between projections of both vertices to \(G\). For the independence number it is well known that

\[
\alpha(G \circ H) = \alpha(G)\alpha(H),
\]

see Theorem 1 in \[3\]. Lexicographic product \(G \circ H\) is connected if and only if \(G\) is connected. For more properties of the lexicographic product see \[7\].

## 2 Results

In this section we present a lower and an upper bound for the packing chromatic number of lexicographic product of graphs. We start with the lower bound and we use the following notation

\[
d(G) = \begin{cases} 
1 & : G \cong K_n \\
\text{diam}(G) - 1 & : \text{otherwise} \end{cases}
\]

**Theorem 2.1.** If \(G\) and \(H\) are graphs, then

\[
\chi_p(G \circ H) \geq |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\text{diam}(G)-1} \rho_i(G) + d(G).
\]

**Proof.** Denote \(\ell = |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\text{diam}(G)-1} \rho_i(G) + d(G)\). Let \(X_1, \ldots, X_k\) be a partition of \(V(G \circ H)\) that yields a \(k\)-packing coloring of \(G \circ H\). We have at most \(\alpha(G)\alpha(H)\) vertices in \(X_1\) by \[2\]. Denote by \(R_i\) a \(\rho_i(G \circ H)\)-set for \(2 \leq i \leq \text{diam}(G) - 1\). By \[1\] we have \(|H^9 \cap R_i| \leq 1\) for every
g \in V(G). So there are at most \( \rho_i(G) \) vertices in \( X_i \) for \( 2 \leq i \leq \text{diam}(G) - 1 \). For \( i \geq \text{diam}(G) \) there can only be one vertex in \( X_i \), since all the vertices are at distance at most \( \text{diam}(G) \) from vertex in \( X_i \). So we have at most \( \alpha(G)\alpha(H) \) vertices colored with color \( 1 \), at most \( \rho_i(G) \) vertices colored with color \( i \) for every \( 2 \leq i \leq \text{diam}(G) - 1 \), and we need one color for each one of the remaining vertices and there are \( |G|\cdot|H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\text{diam}(G)-1} \rho_i(G) \) of them. Meaning that \( \chi_\rho(G \circ H) \geq \ell \) because we have exactly \( d(G) \) color classes which possibly have more than one vertex. 

We continue with an upper bound that has a similar structure as the lower bound from Theorem 2.1.

**Theorem 2.2.** Let \( G \) and \( H \) be graphs and \( k = |H| - \alpha(H) \). If \( H \not\cong \overline{K}_n \), then

\[
\chi_\rho(G \circ H) \leq |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{k+1} \rho_i(G) + k + 1.
\]

**Proof.** Denote \( \ell = |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{k+1} \rho_i(G) + k + 1 \). We know that \( \alpha(G \circ H) = \alpha(G)\alpha(H) \) and it is easy to see that \( \alpha(G \circ H) \)-set can be written as \( A_G \times A_H \) where \( A_G \) is an \( \alpha(G) \)-set and \( A_H \) is an \( \alpha(H) \)-set. We color all the vertices from \( A_G \times A_H \) with color 1. Let \( k = |H| - \alpha(H) \). There remain \( k \) \( G \)-layers with no colored vertices. In each of those layers we color \( \rho_i(G) \) vertices with color \( i \), \( 2 \leq i \leq k + 1 \) (one color \( i \) is used in one layer). Each of the remaining uncolored vertices is colored with its own color. So we have \( \alpha(G)\alpha(H) \) vertices colored with color 1, \( \rho_i(G) \) vertices colored with color \( i \) for every \( 2 \leq i \leq k + 1 \), and we need one color for each one of the remaining uncolored vertices. Clearly, there are \( |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{k+1} \rho_i(G) \) vertices colored with its own color. Meaning that \( \chi_\rho(G \circ H) \leq \ell \) because we have \( k + 1 \) color classes which possibly have more than one vertex. 

Notice that if \( \text{diam}(G) \leq 2 \), then also \( \text{diam}(G \circ H) \leq 2 \) (see [1] and only color 1 can appear more then once in any packing coloring. Therefore, if \( \text{diam}(G) \leq 2 \), Theorem 2.2 also holds for \( H \not\cong \overline{K}_n \). Next we show that if the number of vertices of \( H \) without its \( \alpha(H) \)-set is comparable with \( \text{diam}(G) \), then both bounds coincide.

**Corollary 2.3.** Let \( G \) and \( H \) be graphs and \( |H| - \alpha(H) \geq \text{diam}(G) - 1 \). If \( H \not\cong \overline{K}_n \), then

\[
\chi_\rho(G \circ H) = |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\text{diam}(G)-1} \rho_i(G) + \text{diam}(G) - 1.
\]

**Proof.** Let first \( G \cong K_n \). By Theorem 2.1 it holds that \( \chi_\rho(G \circ H) \geq n|H| - \alpha(H) - \sum_{i=2}^{0} \rho_i(G) + 1 = n|H| - \alpha(H) + 1 \) since \( d(G) = 1 \). On the other hand let \( k = |H| - \alpha(H) \) and we have \( \chi_\rho(G \circ H) \leq n|H| - \alpha(H) - (k + 1 - 2 + 1) + k + 1 = n|H| - \alpha(H) + 1 \) by Theorem 2.2 since \( \rho_i(G) = 1 \) for every \( 2 \leq i \leq k + 1 \). Hence, the equality follows when \( G \cong K_n \). Otherwise \( G \not\cong K_n \) and \( d(G) = \text{diam}(G) - 1 \). So by Theorem 2.1 it holds that \( \chi_\rho(G \circ H) \geq |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\text{diam}(G)-1} \rho_i(G) + \text{diam}(G) - 1 \). Since \( k \geq \text{diam}(G) - 1 \) and \( \rho_i(G) = 1 \) for
every $\text{diam}(G) \leq i \leq k + 1$, by Theorem 2.2 it holds that

$$\chi_\rho(G \circ H) \leq |G| \cdot |H| - \alpha(G)\alpha(H) - \left(\sum_{i=2}^{\text{diam}(G)-1} \rho_i(G) + \sum_{i=\text{diam}(G)}^{k+1} \rho_i(G)\right) + k + 1 =$$

$$= |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\text{diam}(G)-1} \rho_i(G) - (k + 1 - \text{diam}(G) + 1) + k + 1 =$$

$$= |G| \cdot |H| - \alpha(G)\alpha(H) - \sum_{i=2}^{\text{diam}(G)-1} \rho_i(G) + \text{diam}(G) - 1.$$

\[\square\]

We can expect that the condition of Corollary 2.3 will be fulfilled more frequently when $\text{diam}(G)$ is small. In particular, for $\text{diam}(G) = 1$ the condition is always satisfied and we have

$$\chi_\rho(G \circ H) = |G| \cdot |H| - \alpha(H) + 1$$

as seen in the proof of the previous corollary. Notice that in the case of $\text{diam}(G) = 2$ the sum in the lower bound of Theorem 2.1 does not exist and that $d(G) = 1$. Also $\rho_i(G) = 1$ for every $2 \leq i \leq k$ since $\text{diam}(G) = 2$. Therefore we have $- \sum_{i=2}^{k+1} \rho_i(G) + k + 1 = -(k + 1 - 2 + 1) + k + 1 = 1$ in the upper bound of Theorem 2.2. Hence both bounds coincide and we have the following corollary.

**Corollary 2.4.** Let $G$ and $H$ be graphs. If $\text{diam}(G) = 2$, then

$$\chi_\rho(G \circ H) = |G| \cdot |H| - \alpha(G)\alpha(H) + 1.$$

Similar holds also when $\text{diam}(G) = 3$. Namely in this case $\text{diam}(G \circ H) = 3$ by (1) and only two color classes ($X_1$ and $X_2$) can have more than one representative. Therefore bounds from Theorems 2.2 and 2.1 coincide again under condition that there is at least one $G$-layer without vertices from $X_1$. This always occurs if $H \not\iso K_n$ and the following corollary holds.

**Corollary 2.5.** Let $G$ and $H$ be graphs. If $\text{diam}(G) = 3$ and $H \not\iso K_n$, then

$$\chi_\rho(G \circ H) = |G| \cdot |H| - \alpha(G)\alpha(H) - \rho_2(G) + 2.$$

Continuing in this manner things get more complicated. Therefore we finish with an approach from the different side and concentrate on a family of graphs with big diameter, namely the case when $G \iso P_n$. For this we first improve the upper bound from Theorem 2.2.

**Theorem 2.6.** Let $H$ a graph and $n$ a positive integer. If $k = |H| - \alpha(H)$, then

$$\chi_\rho(P_n \circ H) \leq n|H| - \left\lceil \frac{n}{2} \right\rceil \alpha(H) - \sum_{i=2}^{k+1} \left\lceil \frac{n}{i+1} \right\rceil - \sum_{j=k+2}^{\frac{|H|+1}{2}} \left(\left\lceil \frac{n}{j} \right\rceil - 1 \right) + 1 + |H| + 1.$$

\[\text{Proof.}\] Let $P_n = v_1 \ldots v_n$ and $A_H$ be an $\alpha(H)$-set. Clearly, $A_{P_n} = \{v_{2i-1} : i \in \left\lceil \frac{n}{2} \right\rceil\}$ is an $\alpha(P_n)$-set and $A = A_{P_n} \times A_H$ is an $\alpha(P_n \circ H)$-set. Firstly, we color vertices with $k + 1$ colors as in the proof of Theorem 2.2. For this we use

$$\ell = n|H| - \left\lceil \frac{n}{2} \right\rceil \alpha(H) - \sum_{i=2}^{k+1} \left\lceil \frac{n}{i+1} \right\rceil + k + 1$$
colors because \( \rho_i(P_n) = \left\lceil \frac{n}{i+1} \right\rceil \).

In each \( G^h \)-layer, \( h \in A_H \), there exist \( \left\lceil \frac{n}{2} \right\rceil \) still not colored vertices with an even distance between any two of them. We denote them by \( B^h = (V(P_n) - A_{P_n}) \times \{h\} \). Additionally we will color with color \( j \), \( k + 2 \leq j \leq |H| + 1 \), some vertices of exactly one \( G^h \)-layer, \( h \in A_H \). Denote by \( G^h_j \) the \( G^h \)-layer, \( h \in A_H \), containing vertices of color \( j \), \( k + 2 \leq j \leq |H| + 1 \). The biggest distance between two vertices from \( B^h \) equals \( 2 \left\lfloor \frac{n}{2} \right\rfloor - 2 \). Notice that two vertices of \( G^h_j \) colored with \( j \) must be at least \( p_j = 2 \left\lfloor \frac{j}{2} \right\rfloor + 2 \) apart because every second vertex in \( G^h_j \)-layer, \( h \in A_H \), \( k + 2 \leq j \leq |H| + 1 \), is already colored (with color 1). Therefore, we can color with \( j \) vertices from set

\[
\{(v_2 + sp_j, h) : 0 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \}. 
\]

Meaning that \( t_j = \left( \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + 1 \) vertices can be colored with color \( j \), \( k + 2 \leq j \leq |H| + 1 \) in \( G^h_j \)-layer, \( h \in A_H \).

By Theorem \((2.2)\) we use at most \( \ell \) colors for coloring \( P_n \circ H \). In addition \( t_j \) vertices of \( G^h_j \) are colored with \( j \), \( k + 2 \leq j \leq |H| + 1 \). Meaning that

\[
\chi_\rho(P_n \circ H) \leq \ell - \sum_{j=k+2}^{\lfloor |H|+1 \rfloor} t_j + |H| - k
\]

which completes the proof. \( \Box \)

For \( H \cong K_m \) we have \( \alpha(H) = 1 \) and \( k = m - 1 \). The second sum of Theorem \((2.6)\) has only one term and that is in the case of \( j = |H| + 1 \) so we immediately obtain the following.

**Corollary 2.7.** For positive integers \( n \) and \( m \) we have

\[
\chi_\rho(P_n \circ K_m) \leq nm - \left\lceil \frac{n}{2} \right\rceil - \sum_{i=2}^{m} \left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n}{\left\lfloor \frac{m+1}{2} \right\rfloor} \right\rfloor + 1 \} + m.
\]

The upper bound from Theorem \((2.6)\) is not the best possible in the general case which we can see in the example of coloring \( P_8 \circ P_6 \). Using the coloring described in the proof of that theorem we use 32 colors to color \( P_8 \circ P_6 \), see left part of Figure 1. But the same graph can be colored with 31 colors, so \( \chi_\rho(P_8 \circ P_6) \leq 31 \), see right part of Figure 1.

![Figure 1: Packing coloring for \( P_8 \circ P_6 \) using 32 colors according to Theorem \((2.6)\) (a) and 31 colors (b) (not all edges of a graph are drawn).](image-url)
Another example can be constructed as follows. Let $n_t = 1 + \text{lcm}(2, 3, \ldots, t + 1)$, $H \not\cong K_m$ a graph and $k = |H| - \alpha(H)$. Notice that $n_t$ is chosen in such a way that every $\rho_i(P_{n_t})$-set, $1 \leq i \leq t$, contains the first and the last vertex of $P_{n_t}$. If $t - 1 > k$, then we cannot obtain $\rho_i(P_{n_t})$ vertices of color $i$ in $P_{n_t} \circ H$ for some $2 \leq i \leq t$ and the upper bound of Theorem 2.6 is not exact.

References


