Graphs with unique maximum packing of closed neighborhoods

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Abstract

A packing with closed neighborhoods in a graph $G$ is a set $P$ such that any two closed neighborhoods centered in different vertices of $P$ do not intersect. Graphs with unique packing of maximum cardinality are considered. We present several general properties for such graphs which lead to two characterizations of trees with unique maximum packing. One is structural and the other is inductive based on five operations.

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1 Introduction

The packing number of a graph $G$ denoted by $\rho(G)$ is the maximum cardinality of closed neighborhoods that can be packed into a graph such that

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they have pairwise empty intersection. All vertices that are in the centers of mentioned neighborhoods form a packing set. Packing was interesting for a longer period as a natural lower bound for the domination number $\gamma(G)$. One of the first results of that type is from Meir and Moon [13], where it was shown that $\rho(T) = \gamma(T)$ for every tree $T$ (in a different notation). It is easy to see that while the numbers are the same, the sets that yield both $\rho(T)$ and $\gamma(T)$ are often different.

The class of graphs with $\rho(G) = \gamma(G)$, where both maximum packing set and minimum dominating set coincide, is called efficient closed dominated graphs. In such a case we call a minimum dominating set a 1-perfect code. The study of perfect codes in graphs was initiated by Biggs [1]. Later it was intensively studied and we recommend [12] for further informations and references.

In the last decade packing number became more interesting for itself not only in a connection with domination number. The relationship between the packing number and the maximal packing of minimum cardinality, called also the lower packing number, is investigated in [13]. In [14] a connection between packing number and double domination in the form of an upper bound is presented. Graphs for which their packing number equals to the packing number of their complement are described in [3]. In [8] it was shown that the domination number can be also bounded from above by the packing number multiplied with the maximum degree of a graph.

A generalization of packing presented in [5] is $k$-limited packing where every vertex can have at most $k$ neighbors in $k$-limited packing set $S$. A probabilistic approach to $k$-limited packings can be found in [4] to achieve some bounds. A further generalization, that is, generalized limited packing of the $k$-limited packing, see [2], brings a dynamic approach with respect to the vertices of $G$, where $v \in V(G)$ can have a different number of neighbors $k_v$ in a generalized limited packing. The problem is NP-complete, but solvable in polynomial time for $P_4$-tidy graphs as shown in [2].

In this work we are interested in graphs with a special property that there exists only one packing of maximum cardinality. In general one can have many such packings as shown in [11] where an asymptotic bounds for the maximum and the minimum number of packings in a graphs of fixed order are established. On the other hand, we try to present combinatorial properties of graphs with unique maximum packing. One can find sets with several different properties for which this (uniqueness of set with minimum or maximum cardinality) was considered in the literature. For example, see
for graphs with unique maximum independent set, [6] for graphs with unique minimum dominating set and [7] for trees with unique minimum total dominating set. Another approach could also be to count the number of maximum/minimum cardinality sets of a certain type as in [10] for the number of maximum independent sets.

Throughout this work we consider finite undirected simple graphs. Given a vertex \( v \) of a graph \( G \), \( N(v) \) represents the open neighborhood of \( v \), i.e., the set of all neighbors of \( v \) in \( G \) and the degree of \( v \) is \( \deg(v) = |N(v)| \). The closed neighborhood of \( v \in V(G) \) is \( N[v] = N(v) \cup \{v\} \). For any two vertices \( u \) and \( v \), the distance \( d(u, v) \) between \( u \) and \( v \) is the minimum number of edges on a path between \( u \) and \( v \). Given a subset of vertices \( S \) of \( G \), we use \( G - S \) to denote the graph obtained from \( G \) by removing all the vertices from \( S \) and the edges incident with them. If \( S = \{v\} \) for some vertex \( v \), then we simply write \( G - v \). Also, the subgraph of \( G \) induced by \( D \subset V(G) \) will be denoted by \( G[D] \).

A set \( P \subset V(G) \) is a packing of \( G \) if \( d(u, v) > 2 \) for every pair of distinct vertices \( u \) and \( v \) from \( P \). The packing number of \( G \) is the maximum cardinality of any packing of \( G \) and is denoted by \( \rho(G) \). A \( \rho(G) \)-set is a packing of cardinality \( \rho(G) \). If there exists only one maximum packing \( P \) of a graph \( G \), then \( G \) is a graph with a unique \( \rho(G) \)-set.

Let \( G \) be a graph. A leaf of \( G \) is a vertex of degree one. A support vertex of \( G \) is a vertex of degree at least two adjacent to a leaf. A strong support vertex of \( G \) is a support vertex of \( G \) that is adjacent to at least two leaves. Let \( T \) be a tree, \( v \) its arbitrary vertex and \( v_1, \ldots, v_k, k = \deg(v) \), neighbors of \( v \). We denote by \( T_1, \ldots, T_k \) the trees in \( T - v \) such that \( v_i \in T_i \) for every \( i \in \{1, \ldots, k\} \). A root of a tree \( T \) is a special designated vertex of \( T \). Let \( u \) and \( v \) be adjacent in \( T \) such that \( d(u, r) > d(v, r) \) for a root \( r \). In such a case we call \( u \) a down-neighbor of \( v \) and \( v \) is the up-neighbor of \( u \).

## 2 Basic properties and the structure of graphs with unique \( \rho(G) \)-set

We start with several basic properties of graphs with a unique \( \rho(G) \)-set, that will be important later.

**Lemma 2.1.** If a graph \( G \) has a unique \( \rho(G) \)-set \( P \), then every leaf of \( G \) belongs to \( P \).
Proof. Let $P$ be a unique $\rho(G)$-set. To prove the lemma assume there exists a leaf $\ell \notin P$. If its support vertex $x$ is in $P$, then $P' = (P - \{x\}) \cup \{\ell\}$ is a $\rho(G)$-set of $G$ that is different from $P$, which is a contradiction with the assumption. So $x \notin P$. If some neighbor of $x$, say $y$, is in $P$, then $P'' = (P - \{y\}) \cup \{\ell\}$ is a $\rho(G)$-set of $T$ that is different from $P$, the same contradiction again. Thus $N[x] \cap P = \emptyset$. This yields a contradiction with maximum cardinality of $P$ because $P \cup \{\ell\}$ is a packing of bigger cardinality than $P$. Hence, all leaves must be in $P$. \hfill \Box

Lemma 2.2. If $G$ is a graph with a unique $\rho(G)$-set $P$, then $G$ has no strong support vertex.

Proof. Suppose that $G$ has a strong support vertex $v$. That means $v$ has at least two leaves $\ell_1$ and $\ell_2$ as its neighbors. Because all leaves of $G$ belong to $P$ by Lemma 2.1, we have $\ell_1, \ell_2 \in P$, such that $d(\ell_1, \ell_2) = 2$. This is a contradiction with the definition of a packing $P$. \hfill \Box

By the definition, every pair of vertices of a packing must be at distance that is at least three. However, vertices at distance three are obligatory in the case of a unique $\rho(G)$-set.

Lemma 2.3. Let $G$ be a graph on at least two vertices. If $G$ has a unique $\rho(G)$-set $P$, then for every vertex $v \in P$ and his neighbor $v'$ there exists a vertex $u \in P$ such that $d(v, u) = 3$ and $v'$ is on a shortest path between $v$ and $u$.

Proof. Suppose there exists a vertex $v \in P$ and his neighbor $v'$ such that for every vertex $u \in P$ either $d(v, u) > 3$ or $d(v, u) = 3$ and $d(v', u) > 2$. Set $(P - \{v\}) \cup \{v'\}$ is also a $\rho(G)$-set which is a contradiction with the assumption that $P$ is a unique $\rho(G)$-set. \hfill \Box

Notice that in above lemma two or more neighbors of $v \in P$ can be on a shortest path to the same vertex $u \in P$ with $d(v, u) = 3$. A small example for this is presented by a graph $G$ defined on the six-cycle $u_1u_2u_3u_4u_5u_6u_1$ together with edges $u_2u_5$ and $u_3u_6$, where $P = \{u_1, u_4\}$ is the unique $\rho(G)$-set. This changes in the case of trees.

Corollary 2.4. Let $T$ be a tree on at least two vertices. If $T$ has a unique $\rho(T)$-set $P$, then for every vertex $v \in P$ there exist vertices $u_1, \ldots, u_k \in P$, $k = \text{deg}(v)$, such that $d(v, u_i) = 3$ and $u_i \in V(T_i)$ for every $i \in \{1, \ldots, k\}$. 

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Proof. Let $T$ be a tree with a unique $\rho(T)$-set $P$ and $v_1, \ldots, v_k$, $k = \text{deg}(v)$, neighbors of $v \in P$. Suppose that there exists $i \in \{1, \ldots, k\}$ such that no vertex from $V(T_i) \cap P$ is at distance three from $v$ in $T$. Set $P' = (P - \{v\}) \cup \{v_i\}$ is also a $\rho(G)$-set which is a contradiction with the assumption that $P$ is the unique $\rho(G)$-set. \hfill \Box

Above corollary guarantees that in a tree with a unique $\rho(T)$-set $P$, for any $v \in P$ there exists a vertex in $P$ at distance three from $v$ in the direction of each of its neighbors. This is one of the key points in the structural characterization of trees with a unique $\rho(T)$-set, see Theorem 3.1.

Next we present the structure of a graph $G$ with a unique $\rho(G)$-set $P$ (white squared vertices in Figure 1). Set $V(G) - P$ can be partitioned into sets $Q$ (white vertices in Figure 1) and $R$ (black vertices in Figure 1), where $Q$ is the set of all neighbors of vertices from $P$ and $R = V(G) - (P \cup Q)$. Clearly, set $R$ may be empty and $Q$ is empty if and only if $G = \bigcup K_1$. From the definition of a unique $\rho(G)$-set we infer the following properties:

- $P$ is an independent set;
- for every vertex $v \in P$ there exists a vertex $u \in P$ with $d(v, u) = 3$ (by Lemma 2.3);
- every vertex from $Q$ has exactly one neighbor in $P$;
- for every vertex $v' \in Q$ there exists a vertex $v \in P$ with $d(v, v') = 2$ (by Lemma 2.3);
- there are no edges between vertices from $P$ and vertices from $R$;
- every vertex $v$ in $R$ has at least two neighbors in $Q$ (otherwise, if $u$ is the only neighbor of $w$ from $Q$, then is $(P - \{v\}) \cup \{u\}$ also a $\rho(G)$-set, a contradiction);
- $G[Q]$ and $G[R]$ are arbitrary.

We continue with a lemma that describes what must be going on around a vertex from $R$ in a graph with a unique maximum packing.

**Lemma 2.5.** Let $G$ be a graph which has a unique $\rho(G)$-set $P$. If $G$ contains a vertex $v \in R$, then there exist paths $xyzu$ and $x'y'z'u'$ such that $x, x', u, u' \in$
Figure 1: Partition of vertices of a tree $T$ with a unique $\rho(T)$-set $P$.

Let $v$ be a vertex from the set $R$ of a graph $G$. If there is no vertex from $P$ at distance 2 from $v$, then we have a contradiction with the maximality of $P$ since $P \cup \{v\}$ is a packing. Let $x \in P$ be at distance two from $v$ and $y$ a common neighbor of $x$ and $v$. If $\deg(y) = 2$, then $(P - \{x\}) \cup \{y\}$ is also a $\rho(G)$-set, a contradiction with the uniqueness of $P$. So $\deg(y) > 2$. Because $P$ is the unique $\rho(G)$-set, $(P - \{x\}) \cup \{y\}$ is not a packing. This implies that
there exists a vertex \( u \in P \) at distance two from \( y \). Let the common neighbor of \( u \) and \( y \) be \( z \). Clearly \( z \in Q \) and therefore different from \( u \). If \( x \) is the only vertex from \( P \) at distance 2 from \( v \), then \((P - \{x\}) \cup \{v\}\) is also a \( \rho(G) \)-set, a contradiction with the uniqueness of \( P \) again. Therefore there exists \( x' \in P \) so that \( d(v, x') = 2 \) and \( x \neq x' \). Let \( y' \) be a common neighbor of \( v \) and \( x' \). Clearly \( y' \in Q \) and \( y \neq y' \) because otherwise \( d(x, x') \leq 2 \) which is not possible. Assume that there exist no \( z' \in Q \) and \( u' \in P \) such that \( d(v, z') = 2 = d(x', z') \) and \( d(v, u') = 3 = d(x', u') \). In this case \((P - \{x'\}) \cup \{y'\}\) is a \( \rho(G) \)-set which is a contradiction with the uniqueness of \( P \) again.

The possibilities of Lemma 2.5 are presented on Figure 2. Notice that all four graphs must be considered as subgraphs of a bigger graph with a unique \( \rho(G) \)-set \( P \). While first three graphs are itself also graphs with unique \( \rho(G) \)-set \( P \), the last one is not such. The minimum example of a graph that has a unique \( \rho(G) \)-set \( P \) and contains the rightmost graph of Figure 2 as a subgraph can be obtained if we add four additional vertices \( t, t', s, s' \) together with edges \( tz, t'z', z's, ss' \).

3 Trees with unique \( \rho(T) \)-set

In this section we limit ourselves to trees. We present two characterizations of trees with a unique \( \rho(T) \)-set. First is a structural one and is describing properties of the unique \( \rho(T) \)-set. For this, lemmas from the previous sections come in handy. In particular, the only possible outcome of Lemma 2.5 in the case of trees, is the leftmost tree of Figure 2.

**Theorem 3.1.** A tree \( T \) has a unique \( \rho(T) \)-set \( P \) if and only if the following statements are fulfilled for every \( \rho(T) \)-set.

(i) Every leaf is in \( P \).

(ii) For every \( v \in P \) there exists at least one vertex from \( P \) at distance 3 from \( v \) in each \( T_i \), \( 1 \leq i \leq \deg(v) \).

(iii) For every \( v \in R \) there exist different paths \( xyzu \) and \( x'y'z'u' \) such that \( x, x', u, u' \in P \), \( y, y', z, z' \in Q \) and \( vy, vy' \in E(T) \).

**Proof.** If a tree \( T \) has a unique \( \rho(T) \)-set \( P \), then (i) follows from Lemma 2.1, (ii) follows from Corollary 2.4 and (iii) from Lemma 2.5.
To prove the converse suppose that (i), (ii) and (iii) hold for every $\rho(T)$-set. We proceed by induction on the number of vertices $n$ of $T$. If $n = 1$, then $T \cong K_1$ which is the smallest tree with a unique $\rho(T)$-set and the base of the induction is clear. Denote by $T'$ a tree obtained from $T$ by deleting some vertices and by $P'$ its $\rho(T')$-set.

Assume on the way to a contradiction there exist different $\rho(T)$-sets $P_1$ and $P_2$. Let $r \in V(T)$ be a root of $T$. Choose a vertex $t \in (P_1 - P_2) \cup (P_2 - P_1)$ at the maximum distance from $r$. We may assume without loss of generality that $t \in P_1 - P_2$. Vertex $t$ is not a leaf by (i) and there exists a down-neighbor $v$ of $t$. By the choice of $t$ sets $P_1 \cap V(T_v)$ and $P_2 \cap V(T_v)$ must be the same.

Let $T' = T - V(T_v)$ and let $P'_1 = P_1 \cap V(T')$. If $t$ is not a leaf of $T'$, then (i) holds for $T'$ since (i) holds for all the leaves in $T$. Otherwise $t$ is a leaf and $t \in P'$. So (i) holds for $P'$. Clearly (ii) also holds for $P'_1$ in $T'$ since it holds for $P_1$ in $T$. Let $w$ be an arbitrary vertex from $R \cap V(T')$. Notice that $d(t, w) \geq 2$ since $t \in P$. Clearly (iii) holds if $d(t, w) > 2$ since it holds for $P_1$. So let $d(t, w) = 2$ and let $xzu$ and $x'y'z'u'$ be the paths in $T$ such that $x, x', u, u' \in P$, $y, y', z, z' \in Q$ and $vy, vy' \in E(T)$. (Notice that they exist in $T$ by (iii).) Because $t \in P_1$ and $d(t, w) \geq 2$ both mentioned paths must be in $T'$ too, and (iii) holds for $T'$ as well.

By the induction hypothesis $T' = T - V(T_v)$ has a unique $\rho(T')$-set $P'_1$. So $|P'_1| > |P'_2|$, where $P'_2 = P_2 \cap V(T')$. Because $|P_1 \cap V(T_v)| = |P_2 \cap V(T_v)|$ it follows that $|P_1| > |P_2|$ which is a contradiction with $P_2$ being a $\rho(T)$-set and the proof is complete.

The conditions (i), (ii) and (iii) from the last theorem can be checked in polynomial time for a given set $P$. This implies that the problem of recognizing trees with a unique $\rho(T)$-set is in the class $NP$. However it is not clear if there exists a polynomial algorithm for the mentioned problem or not.

We continue with an inductive characterization (which also does not help in algorithmic sense). Let $T'$ be a tree with a unique $\rho(T')$-set $P'$. We introduce five operations to construct from $T'$ a larger tree $T$ with a unique $\rho(T)$-set. We will prove that every tree with unique $\rho(T)$-set can be constructed from $K_1$ with these operations. The operations are illustrated in Figure 3 where we use the notation introduced in Section 2: a vertex from $P'$ is white squared, a vertex from $Q'$ is white circled and a vertex from $R'$ is black circled.

(O1) For $y \in P'$ we obtain $T$ from $T'$ by adding a path $xuv$ and an edge $xy$. 
For $x \in Q'$ we obtain $T$ from $T'$ by adding a path $uv$ and an edge $ux$.

(O₃) For a path $k\ell j z j'k' \ell'$ and vertices $i$ and $i'$ together with edges $ij$ and $i'j'$, where $i, i', \ell, \ell' \in P'$, $j, j', k, k' \in Q'$ and $z \in R'$, we obtain $T$ from $T'$ by adding a path $vuxy$ and an edge $xz$.

(O₄) For a path $v'u'x'y'$, where $u', x' \in Q'$ and $v', y' \in P'$, we obtain $T$ from $T'$ by adding a vertex $z$, a path $vuxy$ and edges $x'z$ and $zx$.

(O₅) For a vertex $w \in R'$ we obtain $T$ from $T'$ by adding a path $vuxzx'w'v'$ and vertices $y$ and $y'$ together with edges $xy$, $x'y'$ and $zw$.

It is easy to see that these five operations are independent. Path $P_{3k+1}$ can be obtained from $K_1$ only by $k$ times applying the operation $O_1$, so operation $O_1$ is needed. The sequence of operations $O_1, O_2$ gives us a tree with a unique maximum packing that cannot be obtained in any other way. Also the sequence $O_1, O_4$ results in the leftmost tree of Figure 2 so operation $O_4$ is needed. In addition, if to the previous sequence we add either $O_3$ to get $O_1, O_4, O_3$ or $O_5$ to get $O_1, O_4, O_5$ we again obtain two trees that cannot be obtained in any other way. Hence we need all five operation and since the mentioned sequences are unique, the operations $O_1 \ldots O_5$ are independent.

**Theorem 3.2.** A tree $T$ has a unique $\rho(T)$-set $P$ if and only if $T$ can be obtained from $K_1$ by a sequence of operations $O_1 \ldots O_5$.

**Proof.** Assume first that $T$ is a tree obtained from $K_1$ by a sequence of operations $O_1 \ldots O_5$. We will show that $T$ is a tree with a unique $\rho(T)$-set by induction on the length $k$ of the mentioned sequence. If $k = 0$, then $T \cong K_1$ which is a tree with a unique $\rho(T)$-set. Let now $k > 0$ and let $T'$ be a tree obtained from $K_1$ by using the same sequence as for $T$, but without including the last step. By the induction hypothesis, $T'$ is a tree with a unique $\rho(T')$-set $P'$. We will use the notation presented in Figure 3.

Let $T$ be a tree obtained from $T'$ by operation $O_1$. Clearly $P = P' \cup \{v\}$ is a maximum packing of $T$ because $P'$ is a maximum packing of $T'$. Let $P_1$ be a packing of $T$ such that either $u \in P_1$ or $x \in P_1$. We have $P_1 \cap V(T') \neq P'$, because $y \notin P_1$. By uniqueness of $P'$ we have $|P_1 \cap V(T')| < |P'|$. Therefore $|P_1| < |P|$ and $P_1$ is not a $\rho(T)$-set. Hence $P$ is the unique $\rho(T)$-set.

Assume now $T$ is obtained from $T'$ by operation $O_2$. We will prove that $T$ has a unique $\rho(T)$-set $P = P' \cup \{v\}$. Because $x \in Q$ there exists $y \in P'$ which is a neighbor of $x$. Let $P_1$ be a packing of $T$ such that $u \in P_1$.
Clearly $P_1 \cap V(T') \neq P'$, because $y \notin P_1$. By uniqueness of $P'$ we have $|P_1 \cap V(T')| < |P'|$. Therefore $|P_1| < |P|$ and $P_1$ is not a $\rho(T)$-set. Hence $P$ is the unique $\rho(T)$-set.

Suppose next that we apply operation $O_3$ on $T'$ to get $T$. If $u \in P$ (or $x \in P$), then $P' \cup \{u\}$ (or $P' \cup \{x\}$) has $|P'| + 1$ elements. But setting $P = P' \cup \{v,y\}$ we get a packing of $T$ with $|P'| + 2$ elements, so $P' \cup \{u\}$ (or $P' \cup \{x\}$) is not a $\rho(T)$-set. Notice also that every packing $P_1$ of $T$ with $|P_1 \cap V(T')| < |P'|$ has less than $|P'| + 2$ elements. Meaning that $T$ has a unique $\rho(T)$-set $P = P' \cup \{v,y\}$.

If operation $O_4$ is applied to get $T$ from $T'$, then $u$ and $x$ do not belong to a $\rho(T)$-set by the same argument as in the case of $O_3$. Suppose that there exists a packing $P_1$ of $T$ such that $z \in P_1$. Clearly, $y,y' \notin P_1$ and $|P_1 \cap V(T')| < |P'|$ because $P'$ is the unique maximum packing of $T'$. This implies that $|P_1| < |P|$ for $P = P' \cup \{v,y\}$. Therefore, $P_1$ is not a maximum packing of $T$ and $P$ is the unique $\rho(T)$-set.
Finally suppose that $T$ is obtained from $T'$ by operation $O_5$. If a packing of $T'$ contains $x$ (or $x'$), then $P' \cup \{x, y', v'\}$ (or $P' \cup \{x', y, v\}$) has $|P'| + 3$ elements and no packing of $T$ that contains $x$ (or $x'$) contains more elements. Similarly, if $u \in P$ (or $u' \in P'$), then $P' \cup \{u, y', v'\}$ (or $P' \cup \{u', y, v\}$) contains $|P'| + 3$ elements, which is again best possible in this case. If $z \in P$, then $P' \cup \{z, v, v'\}$ has also $|P'| + 3$ elements and more is not possible. But setting $P = P' \cup \{v, y, v', y'\}$ we get a packing of $T$ with $|P'| + 4$ elements, so any packing with $|P'| + 3$ elements or less is not a $\rho(T)$-set. Meaning that $T$ has a unique $\rho(T)$-set $P = P' \cup \{v, y, v', y'\}$.

To prove the converse, let $T$ be a tree with a unique $\rho(T)$-set $P$. Let $r \in V(T)$ be a vertex of $T$ and consider $T$ as a rooted tree with the root $r$. Let $v$ be a leaf of $T$ that is at the maximum distance from $r$. We proceed by induction on the number of vertices of $T$. If $v = r$, then $T \cong K_1$ which is the smallest tree with a unique $\rho(T)$-set, hence the base of the induction is clear. Let $v \neq r$ and let $u$ be the support vertex of $v$. Clearly $u \neq r$ because $T$ has a unique $\rho(T)$-set. Since $v$ is a leaf of $T$ that is at the maximum distance from $r$ we know that $\deg(u) = 2$ by Lemma 2.2. Let $x$ be an up-neighbor of $u$. Clearly $x \notin P$ because $d(x, v) = 2$. Assume $x \in R$. By Lemma 2.5 we immediately get a contradiction since $v$ is a leaf at the biggest distance from $r$. Therefore $x \in Q$. Let $y$ be a neighbor of $x$ that is in $P$, $z$ be the up-neighbor of $x$ and $w$ be the up-neighbor of $z$ (if it exists). If $x$ has a neighbor in $R$, then it must be the up-neighbor of $x$ by Lemma 2.5 again. Denote by $T'$ a tree obtained from $T$ by deleting some vertices and by $P'$ its $\rho(T')$-set. We distinguish the following cases.

**Case 1:** $z \notin R$

**Subcase 1.1:** $\deg(x) = 2$.
Notice that in this case $z = y$. We obtain a tree $T'$ from $T$ by deleting vertices $x$, $u$ and $v$. Assume that $T'$ has two $\rho(T')$-sets $P_1$ and $P_2$. Then $P_1 \cup \{v\}$ and $P_2 \cup \{v\}$ are both $\rho(T)$-sets which is a contradiction with the uniqueness of the $\rho(T)$-set. So $T'$ has a unique $\rho(T')$-set. By the induction hypothesis $T'$ can be built from $K_1$ by a sequence of operations $O_1 - O_5$. If we add the operation $O_1$ at the end of this sequence, then we obtain $T$ from $K_1$ by a sequence of operations $O_1 - O_5$.

**Subcase 1.2:** $\deg(x) \geq 3$.
Let $z_1, \ldots, z_k$ be down-neighbors of $x$ (different from $y$ in the case that $y$ is a down neighbor of $x$). Again every $z_i$, $1 \leq i \leq k$, is not in $R$ by Lemma 2.5.
and the choice of $v$ and is therefore in $Q$. Notice that every down neighbor
of $z_i$, $1 \leq i \leq k$, must be a leaf by the choice of $v$. By Lemma 2.2 every $z_i$,
$1 \leq i \leq k$ has exactly one down-neighbor $w_i$ which is in $P$ since $y \in P$.
If $z = y$, we obtain a tree $T'$ from $T$ by deleting a subtree rooted by $x$.
If there exist two $\rho(T')$-sets $P_1$ and $P_2$ with $|P| - 1 - k$ elements, then
$P_1 \cup \{v, w_1, \ldots, w_k\}$ and $P_2 \cup \{v, w_1, \ldots, w_k\}$ are both $\rho(T)$-sets which is a
contradiction with $\rho(T)$-set being unique. So $T'$ has a unique $\rho(T')$-set. By
the induction hypothesis $T'$ can be built from $K_1$ by a sequence of operations
$O_1 - O_5$. If we add the operation $O_1$ for $x$, $u$ and $v$ and $k$ times operation
$O_2$ for $z_i$ and $w_i$, $1 \leq i \leq k$, at the end of this sequence, then we obtain $T$
from $K_1$ by a sequence of operations $O_1 - O_5$.

If $z \neq y$ (so $y$ is a down-neighbor of $x$), then $z \in Q$ as $d(y, z) = 2$. So $z$
has one neighbor $t$ which is in $P$. Denote with $a_1, \ldots, a_{\ell}$ down-neighbors of $z$
different from $t$ (if $t$ is a down neighbor). Every $a_i$, $1 \leq i \leq \ell$, is not in $R$ by
Lemma 2.5 and by the choice of $v$. Also every $a_i$, $1 \leq i \leq \ell$ is not in $P$ because
d($t, a_i$) = 2 and must therefore be in $Q$. Every vertex from $Q$ has exactly
one neighbor in $P$ and let $b_i$ be such a neighbor of $a_i$, $1 \leq i \leq \ell$. Because $z$
is not in $P$, $b_i$ is a down-neighbor of $a_i$. In addition every $a_i$, $1 \leq i \leq \ell$, can
have $c_{i, 1}, \ldots, c_{i, j_i}$ down-neighbors which must be in $Q$ (they cannot be in $R$
by Lemma 2.5 and are not in $P$ as $d(c_{i, s}, b_i) = 2$ for $1 \leq s \leq j_i$). Furthermore,
every $c_{i, s}$, $1 \leq i \leq \ell$ and $1 \leq s \leq j_i$, has exactly one down-neighbor
d$_{i, s}$ by the choice of $v$ and by Lemma 2.2 and $d_{i, s}$ must be in $P$ by Lemma 2.1.

If $w = t$ we obtain a tree $T'$ from $T$ by deleting a subtree rooted by $z$.
If there exist two $\rho(T')$-sets $P_1$ and $P_2$ with $|P| - 2 - k - \ell - j_1 - \cdots - j_\ell$
elements, then for

$$A = \{v, y, w_1, \ldots, w_k, b_1, \ldots, b_\ell, d_{1, 1}, \ldots, d_{1, j_1}, \ldots, d_{\ell, 1}, \ldots, d_{\ell, j_\ell}\}$$

sets $P_1 \cup A$ and $P_2 \cup A$ are both $\rho(T)$-sets which is a contradiction with
$\rho(T)$-set being unique. So $T'$ has a unique $\rho(T')$-set. By the induction hy-
pothesis $T'$ can be build from $K_1$ by a sequence of operations $O_1 - O_5$. If
we add the operation $O_1$ for $z$, $x$ and $y$ and operation $O_2$ once for $u$ and
$v$, $k$ times for $z_i$ and $w_i$, $1 \leq i \leq k$, $\ell$ times for $a_i$ and $b_i$, $1 \leq i \leq \ell$, and
$j_1 + \cdots + j_\ell$ times for $c_{i, q}$ and $d_{i, q}$, $1 \leq i \leq \ell$ and $1 \leq q \leq j_i$, at the end of
this sequence, then we obtain $T$ from $K_1$ by a sequence of operations $O_1 - O_5$. 12
If \( w \neq t \) (so \( t \) is a down-neighbor of \( z \)) we obtain a tree \( T' \) from \( T \) by deleting vertices \( u \) and \( v \). Notice that no \( \rho(T') \)-set contains \( x \) since \( (P' - \{x\}) \cup \{y, t, w_1, \ldots, w_k\} \) would be a packing of cardinality greater than \( \rho(T') \)-set. If there exist two \( \rho(T') \)-sets \( P_1 \) and \( P_2 \) with \( |P| - 1 \) elements, then \( P_1 \cup \{v\} \) and \( P_2 \cup \{v\} \) are both \( \rho(T) \)-sets which is a contradiction with \( \rho(T) \)-set being unique. So \( T' \) has a unique \( \rho(T') \)-set. By the induction hypothesis \( T' \) can be built from \( K_1 \) by a sequence of operations \( O_1 - O_5 \). If we add the operation \( O_2 \) for \( u \) and \( v \) at the end of this sequence, then we obtain \( T \) from \( K_1 \) by a sequence of operations \( O_1 - O_5 \).

**Case 2: \( z \in R \)**

In this case there exists \( y \in P \) that is a down-neighbor of \( x \) otherwise also \( (P - \{v\}) \cup \{u\} \) is a \( \rho(T) \)-set which is not possible. Denote by \( z_1, \ldots, z_k \) down-neighbors of \( x \) (if they exist) different from \( y \). Clearly every \( z_i, 1 \leq i \leq k \), is in \( Q \) as they cannot be in \( R \) by Lemma 2.5 nor in \( P \) because \( d(z_i, y) = 2 \). By Lemma 2.1 every \( z_i, 1 \leq i \leq k \), has a down-neighbor \( w_i \) which is in \( P \). By the choice of \( v \) and by Lemma 2.2, \( w_i \) is the unique down-neighbor of \( z_i \). Let \( a_1, \ldots, a_\ell \) be down-neighbors of \( z \) (if they exist). Clearly every \( a_i, 1 \leq i \leq \ell \), is in \( Q \) because \( a_i \) cannot be in \( R \) by Lemma 2.5 and the choice of \( v \) and not in \( P \) as neighbors of \( z \). Every \( a_i, 1 \leq i \leq \ell \), has exactly one down-neighbor \( t_i \) which is in \( P \). All the other down-neighbors of \( a_i \) are in \( Q \) because they are at distance two from \( t_i \) and therefore not in \( P \) and not in \( R \) by the choice of \( v \) and by Lemma 2.5. We denote them by \( b_{a_1,j}, 1 \leq i \leq \ell, 1 \leq j \leq m_i \). Every \( b_{a_1,j}, 1 \leq i \leq \ell, 1 \leq j \leq m_i \), has exactly one down-neighbor \( c_{a_1,j} \in P \). By the choice of \( v \) and by Lemma 2.2, \( c_{a_1,j} \) is the unique down-neighbor of \( b_{a_1,j} \). Notice that by Lemma 2.3 for \( t_i \) and \( a_i \) we have \( m_i \geq 1 \) for every \( 1 \leq i \leq \ell \).

We will use the following notation

\[
A = \{v, y, w_1, \ldots, w_k, t_1, \ldots, t_\ell, c_{a_1,1}, \ldots, c_{a_1,m_1}, \ldots, c_{a_\ell,1}, \ldots, c_{a_\ell,m_\ell}\}.
\]

**Subcase 2.1: \( \deg(z) = 2 \).**

Notice that in this case vertices \( a_i, 1 \leq i \leq \ell \) do not exist whenever \( z \neq r \) and if \( z = r \), then \( \ell = 1 \). We obtain a tree \( T' \) from \( T \) by deleting a subtree rooted by \( z \). Suppose \( T' \) has two \( \rho(T') \)-sets \( P_1 \) and \( P_2 \). In that case \( P_1 \cup \{v, y, w_1, \ldots, w_k\} \) and \( P_2 \cup \{v, y, w_1, \ldots, w_k\} \) are both \( \rho(T) \)-sets which is a contradiction with the uniqueness of \( \rho(T) \)-set. Meaning that \( T' \) has a unique \( \rho(T') \)-set. By the induction hypothesis \( T' \) can be built from \( K_1 \) by a sequence of operations \( O_1 - O_5 \). If we add the operation \( O_4 \) for vertices
z, y, x, u, v at the end of this sequence and then continue with \( k \) times adding operation \( O_2 \) for \( z_i \) and \( w_i \), \( 1 \leq i \leq k \), we obtain \( T \) from \( K_1 \) by a sequence of operations \( O_1 - O_5 \).

**Subcase 2.2:** \( \text{deg}(z) \geq 3 \) and \( z \) does not have any neighbors in \( R \).
Clearly \( w \in Q \), since \( z \) does not have any neighbors in \( R \). So \( w \) has a neighbor \( s_1 \in P \). By Lemma 2.3, there exists a neighbor \( s_2 \in Q \) of \( w \) and a neighbor \( s_3 \in P \) of \( s_2 \).

We obtain a tree \( T' \) from \( T \) by deleting a subtree rooted by \( z \). If there exist two different \( \rho(T') \)-sets \( P_1 \) and \( P_2 \) with \( |P| - 2 - k - \ell - m_1 - \cdots - m_\ell \) elements, then sets \( P_1 \cup A \) and \( P_2 \cup A \) are both \( \rho(T) \)-sets which is a contradiction with \( \rho(T) \)-set being unique. So \( T' \) has a unique \( \rho(T') \)-set. By the induction hypothesis \( T' \) can be built from \( K_1 \) by a sequence of operations \( O_1 - O_5 \). If we add the operation \( O_4 \) for \( z, x, y, u, v, \ell \) times operation \( O_3 \) for \( a_i, t_i, b_{a_i,1}, c_{a_i,1}, 1 \leq i \leq \ell, m_1 + \cdots + m_\ell - \ell \) times operation \( O_2 \) for \( b_{a_i,r}, c_{a_i,r}, 1 \leq i \leq \ell, 2 \leq r \leq m_i \), and \( k \) times operation \( O_2 \) for \( z_i, w_i, 1 \leq i \leq k \), at the end of this sequence, then we obtain \( T \) from \( K_1 \) by a sequence of operations \( O_1 - O_5 \).

**Subcase 2.3:** \( \text{deg}(z) \geq 3 \) and \( z \) does have a neighbor in \( R \).
By Lemma 2.5 and the choice of \( v \) vertex \( w \) is in the set \( R \). Tree \( T' \) is obtained from \( T \) by deleting a subtree rooted by \( z \). By Lemma 2.5, \( z \) has at least one down-neighbor different from \( x \), which means that \( \ell \geq 1 \). Suppose \( T' \) has two different \( \rho(T') \)-sets \( P_1 \) and \( P_2 \). In that case \( P_1 \cup A \) and \( P_2 \cup A \) are both \( \rho(T) \)-sets which is a contradiction with the uniqueness of \( \rho(T) \)-set. So \( T' \) has a unique \( \rho(T') \)-set. By the induction hypothesis \( T' \) can be built from \( K_1 \) by a sequence of operations \( O_1 - O_5 \). If we add the operation \( O_5 \) for \( z, x, y, u, v, a_1, t_1, b_{a_1,1}, c_{a_1,1}, \ell - 1 \) times operation \( O_3 \) for \( a_i, t_i, b_{a_i,1}, c_{a_i,1}, 2 \leq i \leq \ell, m_1 + \cdots + m_\ell - \ell \) times operation \( O_2 \) for \( b_{a_i,r}, c_{a_i,r}, 1 \leq i \leq \ell, 2 \leq r \leq m_i \), and \( k \) times operation \( O_2 \) for \( z_i, w_i, 1 \leq i \leq k \), at the end of this sequence, then we obtain \( T \) from \( K_1 \) by a sequence of operations \( O_1 - O_5 \). \( \square \)

**References**


