The median game*

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Abstract

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We introduce a game which is played by two players on a connected graph. The players I and II alternately choose vertices of the graph until all vertices are taken. The set of vertices chosen by player I is denoted by Π_I, and by II is denoted by Π_{II}. The objective of player I (resp. II) is to minimize the sum of distances of a vertex in Π_I (resp. Π_{II}) to all the other vertices in Π_I (resp. Π_{II}). We give a necessary condition for a tree so that player I (who begins the game) has a winning strategy for the game. We prove also that for hypercubes and some other symmetric graphs the player II has a strategy to draw the game. Complete bipartite graphs are considered as well.

**Key words:** median game, location theory, trees

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1 Introduction

Many problems in combinatorics have their origins in games. For example the famous problem of Hamiltonicity of a graph started as a game on the octahedron. The tower of Hanoi problem also started as a game where the player should arrange the discs in a prescribed order (see [12] for a comprehensive historical and mathematical insight). Also the motivation for the first publication in graph theory by Euler may be considered as a game, where the players objective is to find a way over all Königsberg bridges and return to the origin.

All games mentioned above are played by one person, however the standard notion of a game in game theory involves two players opposing each other in order to meet the objective of the game. One such game is the coloring game introduced by Gardner in [7], where the main problem is to determine the game chromatic number of a graph. This question was asked for many different classes of graphs, and in particular it was proved in [19] that the game chromatic number of a planar graph is at most 17.

Similar are the marking game and the greedy game. Both games were introduced by Zhu (see [18, 10]), the marking game in context of studying the game chromatic number and the greedy game as an intermediate between the marking and the coloring game. There are also other well studied games on graphs, such as the cop and robber game (see [1, 15, 8]) and the domination game (see [6, 11]).

The problem of finding an optimal location for placing a facility is a classical problem in optimization. The theoretical and algorithmical aspects of this problem are extensively studied in the literature, see [16, 17, 13]. The problem of locating the distribution center is conventionally modelled as the median location problem and the problem of locating median sets for profiles on graphs was studied by many authors with different assumptions (see, for example, [3, 5, 4, 14]). In this paper we introduce a new concept where the main objective is to choose several locations so that certain distance-related conditions are fulfilled. The concept introduced in this paper is motivated by the following situation.

Suppose there are two companies competing for a finite set of locations, and assume that the distance between any two locations from this set is known. Each company can
buy one location at a time, and they alternatively buy locations until all locations are sold. The objective of each company is to get such locations so that the total distance from one of the locations (owned by the company), where the distribution center of the company is going to be located, to all the other locations owned by the company is as small as possible. We model this problem by a game on a graph, and call it the median game.

Similar model was introduced in [2], where one has a distribution center on every location. Hence, the objective is to minimize the sum of distances between all pairs of locations owned by the company. This game is called the Wiener game in [2].

Let $G = (V, E)$ be a graph and $\pi \subseteq V$ be a set, which we call a profile. The remotness (total distance) of a vertex $x$ with respect to $\pi$ is defined as $R(x) = \sum_{y \in \pi} d(x, y)$. The median value of $\pi$, denoted as $M(\pi)$, is the minimum remotness of a vertex in $\pi$ with respect to $\pi$. If $\pi = V(G)$, then $M(\pi)$ is called a median vertex of $G$. The median game is played by two players $I$ and $II$ on a graph $G$. Player $I$ starts and takes a vertex of $G$, then player $II$ takes another vertex. The game continues this way until all vertices of $G$ are taken. Let $\pi_I$ and $\pi_{II}$ denote the set of vertices that belong to players $I$ and $II$ at the end of the game, respectively. The winner of the game is the player with smaller median value of his/her profile. That is, if $M(\pi_I) < M(\pi_{II})$, player $I$ wins and if $M(\pi_{II}) < M(\pi_I)$, player $II$ wins. Otherwise it is a draw. We use $x_i$ and $y_i$ to denote the vertex taken by player $I$ and $II$ in the $i$-th turn, respectively.

With this we can define a new graph invariant median game number denoted by $\mu(G)$ which is simply $\mu(G) = M(\pi_{II}) - M(\pi_I)$ when both players are playing optimally. With this terminology we have $\mu(G) > 0$ whenever player $I$ has a winning strategy, $\mu(G) < 0$ whenever player $II$ has a winning strategy and $\mu(G) = 0$ if no player has a winning strategy.

The order of a graph seems to play an important role for this game. Namely, if $G$ has an even number of vertices, then both players obtain the same number of vertices, but if $G$ is odd, then player $I$ gets one vertex more than player $II$. Despite this fact, we cannot say that median game is more 'fair' for graphs of even order, since player $I$ can choose at the beginning the vertex which will be the median of his profile and get with this a big advantage.

2 Results

Since player $I$ starts the game it has an obvious advantage and there are numerous graphs for which player $I$ has a strategy to win the median game. Even more, the difference between $M(\pi_I)$ and $M(\pi_{II})$ may be arbitrary large. An example of such graph is a star on an even number $n$ of vertices, where player $I$ takes the central vertex, so his median value is $n/2 - 1$, whereas the player $II$ has median value $n - 2$, because this value is the distance of his profile from a pendant vertex. If $n$ is odd and
player I occupies central vertex, then $M(\pi_I) = \frac{n-1}{2}$ and $M(\pi_{II}) = n - 1$. Altogether we have

$$\mu(K_{1,n-1}) = \left\lfloor \frac{n-1}{2} \right\rfloor.$$ 

and I wins for every $n > 2$. On the other hand we are not aware of any graph of even order for which player II has a strategy to win the game.

**Question 2.1** Is there a graph $G$ of even order such that player II has a strategy to win the median game on $G$?

There are numerous examples of graphs of even order for which the player II has a drawing strategy. These graphs may roughly be described as symmetric graphs, such as even cycles, even paths, hypercubes, and complete bipartite graphs with equal parts. We give a more precise description of graphs that admit a drawing strategy for player II in the following theorem.

**Theorem 2.2** Let $G$ be a graph. If there is an automorphism $\phi : V(G) \to V(G)$ such that $\phi$ has no fixed points and $\phi^2 = id$, then $\mu(G) = 0$.

**Proof.** First note that, since an automorphism $\phi$ has no fixed vertices, graph $G$ has an even order. Next we show that player I cannot win. If player II choose to take vertex $y_i = \phi(x_i)$, then for any fixed vertex $x_i \in \pi_I$ and $y_i = \phi(x_i)$, we have

$$\sum_{j=1}^{\frac{|V(G)|}{2}} d(x_i, x_j) = \sum_{j=1}^{\frac{|V(G)|}{2}} d(\phi(x_i), \phi(x_j)) = \sum_{j=1}^{\frac{|V(G)|}{2}} d(y_i, y_j).$$

Hence I is not a winner on $G$.

On the other hand if player II does not follow the drawing strategy described above, then there is a turn where player I chooses a vertex $x$ and player II does not choose $\phi(x)$. Now player I starts to follow player II by choosing $\phi(y)$, where $y$ is the vertex taken by player II in his last turn. Player I follows this until player II chose $\phi(x)$. Afterwards player I repeat this strategy. Again we have

$$\sum_{j=1}^{\frac{|V(G)|}{2}} d(x_i, x_j) = \sum_{j=1}^{\frac{|V(G)|}{2}} d(\phi(x_i), \phi(x_j)) = \sum_{j=1}^{\frac{|V(G)|}{2}} d(y_i, y_j)$$

and II is not a winner on $G$. Thus game ends in a draw and we have $\mu(G) = 0$. □

There are many examples of graphs which admit an automorphism described in Theorem 2.2. Moreover, if a graph $G$ has such an automorphism, then also $G \square H$, $G \boxtimes H$, $G \times H$, $G \circ H$ and $H \circ G$ admit such an automorphism, where $G \square H$ denotes the Cartesian product, $G \boxtimes H$ the strong product, $G \times H$ the direct product and $G \circ H$ the lexicographic product. (For the definition and properties of graph products see [9].) In particular, hypercube $Q_n$ is a special example of Cartesian product of $n$ factors of $K_2$. Whence we have the following.
Corollary 2.3 If \( n \) is a positive integer, then \( \mu(Q_n) = 0 \).

It is natural to expect that the optimal strategy for player \( I \) is to start with a median vertex of the graph, because this vertex minimizes the sum of distances to all other vertices of the graph. In most cases that we observed starting with the median vertex might be extended to a strategy that leads to a win or a draw in the median game. However Proposition 2.4 proves that the median vertex is not always a good vertex to start with.

Proposition 2.4 For two integers \( m \geq n > 1 \) we have that

\[
\mu(K_{m,n}) = \begin{cases} 
1 & : \ m \neq n \text{ and } m \text{ and } n \text{ are odd} \\
0 & : \ m = n \text{ or } m \text{ and } n \text{ are even} \\
-1 & : \ m \text{ is odd and } n \text{ is even} \\
-2 & : \ m \text{ is even and } n \text{ is odd}
\end{cases}
\]

Proof. Let \( X,Y \) be the partition of \( V(K_{m,n}) \) on independent sets, and assume that \( m = |X| \geq |Y| = n > 1 \). If \( m = n \), then we can use Theorem 2.2 and we have \( \mu(K_{m,n}) = 0 \). If both \( m \) and \( n \) are even, then player \( II \) can copy the move from player \( I \) in the same part \( X \) or \( Y \): if \( x_i \) is in \( X \), then \( II \) follows with \( y_i \) in \( X \) and if \( x_i \) is in \( Y \), then also \( y_i \) is in \( Y \). With this we have \( \mu(K_{m,n}) \leq 0 \). The strategy of \( I \) to get \( \mu(K_{m,n}) \geq 0 \) is as follows. Player \( I \) follows with \( x_{i+1} \) to \( y_i \) of player \( II \) in the same part \( X \) or \( Y \) until it is possible. The distribution of both profiles \( \pi_I \) and \( \pi_{II} \) is symmetric in \( X \) and in \( Y \), which results in \( \mu(K_{m,n}) = 0 \).

Suppose now that \( m \) and \( n \) are odd, and that \( m \neq n \). The strategy of player \( I \) is as follows: if there is exactly one free vertex in \( Y \) then choose this vertex. Otherwise choose a vertex in \( X \), if there are any free vertices in \( X \), and a vertex in \( Y \) if there are no free vertices in \( X \). (We call this strategy at least one in \( Y \) strategy.) With this strategy \( |\pi_I \cap X| > |\pi_{II} \cap X| \) and \( |\pi_I \cap Y| < |\pi_{II} \cap Y| \). Therefore, for a vertex \( x \in Y \cap \pi_I \) we have

\[
M(x) = |\pi_I \cap X| + 2(|\pi_I \cap Y| - 1) = |\pi_I| + |\pi_I \cap Y| - 2,
\]

and for a \( y \in Y \cap \pi_{II} \) we have

\[
M(y) = |\pi_{II} \cap X| + 2(|\pi_{II} \cap Y| - 1) = |\pi_{II}| + |\pi_{II} \cap Y| - 2.
\]

Since \( |\pi_I \cap X| + |\pi_I \cap Y| = |\pi_{II} \cap X| + |\pi_{II} \cap Y| \), we find that \( M(x) < M(y) \). If \( y \in X \cap \pi_{II} \), then \( M(y) = |\pi_{II} \cap Y| + 2(|\pi_{II} \cap X| - 1) \). Since \( |\pi_{II} \cap X| > |\pi_I \cap Y| \), we have again \( M(x) < M(y) \). Altogether we have \( \mu(K_{m,n}) > 0 \). On the other hand, if player \( II \) also follows the at least one in \( Y \) strategy, then we get

\[
|\pi_I \cap X| = |\pi_{II} \cap X| + 1 \text{ and } |\pi_I \cap Y| = |\pi_{II} \cap Y| - 1.
\]  

(1)

It is easy to see that \( \pi_I \cap Y \neq \emptyset \) and that \( \pi_{II} \cap Y \neq \emptyset \). Moreover, any vertex \( x \) from \( \pi_I \cap Y \) is a median vertex for \( \pi_I \) and any vertex \( y \) from \( \pi_{II} \cap Y \) is a median vertex for \( \pi_{II} \). By (1) we get \( M(x) = M(y) + 1 \) and \( \mu(K_{m,n}) = 1 \).
Let now $m$ be odd and let $n$ be even. Note that $n + m$ is odd now, which results in $|\pi_I| = |\pi_{II}| + 1$. If player $II$ follows at least one in $Y$ strategy, then he will achieve that $|\pi_I \cap Y| \geq |\pi_{II} \cap Y|$. For a vertex $x \in Y \cap \pi_I$ we have again $M(x) = |\pi_I| + |\pi_I \cap Y| - 2$, and for a $y \in Y \cap \pi_{II}$ we have $M(y) = |\pi_{II}| + |\pi_{II} \cap Y| - 2$. Thus we find that $M(x) > M(y)$. If $x \in X \cap \pi_I$, then $M(x) = |\pi_I| + |\pi_I \cap X| - 2$. By the at least one in $Y$ strategy for player $II$ we have $|\pi_I \cap X| \geq |\pi_{II} \cap Y|$, which results in $M(x) > M(y)$ again. Altogether we have $\mu(K_{m,n}) < 0$. On the other hand, if player $I$ also follows the at least one in $Y$ strategy, then we have $|\pi_I \cap Y| \leq |\pi_{II} \cap Y|$. This inequality yields

$$|\pi_{II}| + 1 + |\pi_I \cap Y| - 2 \leq |\pi_{II}| + |\pi_{II} \cap Y| - 2 + 1,$$

which gives that $M(x) \leq M(y) + 1$ for $x \in Y \cap \pi_I$ and $y \in Y \cap \pi_{II}$. By the at least one in $Y$ strategy for player $I$, we have $|\pi_I \cap X| \leq |\pi_{II} \cap Y| + 1$. This inequality results in

$$|\pi_{II}| + 1 + |\pi_I \cap X| - 2 \leq |\pi_{II}| + |\pi_{II} \cap Y| - 2 + 1,$$

which also gives $M(x) \leq M(y) + 1$. Altogether we have that $M(x) = M(y) + 1$ and thus $\mu(K_{m,n}) = -1$.

Finally, let $m$ be even and let $n$ be odd. Again $n + m$ is odd and we have $|\pi_I| = |\pi_{II}| + 1$. If player $II$ follows at least one in $Y$ strategy, then he will achieve that $|\pi_I \cap Y| > |\pi_{II} \cap Y|$. Therefore we have

$$|\pi_{II}| + 1 + |\pi_I \cap Y| - 2 > |\pi_{II}| + |\pi_{II} \cap Y| - 2 + 1.$$

For a vertex $x \in Y \cap \pi_I$ and for a $y \in Y \cap \pi_{II}$ this yields $M(x) \geq M(y) + 2$. Also by the at least one in $Y$ strategy for player $II$ we have $|\pi_I \cap X| \geq |\pi_{II} \cap Y| + 1$, which gives

$$|\pi_{II}| + 1 + |\pi_I \cap X| - 2 > |\pi_{II}| + |\pi_{II} \cap Y| - 2 + 2$$

for $x \in X \cap \pi_I$ and $y \in Y \cap \pi_{II}$. Again $M(x) \geq M(y) + 2$ follows and we have $M(I) \geq M(II) + 2$ and $\mu(K_{m,n}) \leq -2$. Let now player $I$ follow the at least one in $Y$ strategy. We have $|\pi_I \cap Y| \leq |\pi_{II} \cap Y| + 1$, which results in

$$|\pi_{II}| + 1 + |\pi_I \cap Y| - 2 \leq |\pi_{II}| + |\pi_{II} \cap Y| - 2 + 2.$$

This implies that $M(x) \leq M(y) + 2$ for $x \in Y \cap \pi_I$ and $y \in Y \cap \pi_{II}$. By the at least one in $Y$ strategy for player $I$, we also have $|\pi_I \cap X| \leq |\pi_{II} \cap Y| + 1$. This leads to

$$|\pi_{II}| + 1 + |\pi_I \cap X| - 2 \leq |\pi_{II}| + |\pi_{II} \cap Y| - 2 + 2,$$

which also gives $M(x) \leq M(y) + 2$. Altogether we have that $M(x) = M(y) + 2$ and thus $M(I) = M(II) + 2$, which gives $\mu(K_{m,n}) = -2$ and the proof is complete. \hfill \Box

Lemma 2.5 Let $G$ be a graph of even order and $\pi \subseteq V(G)$ be a profile in $G$. Suppose that $m \in V(G)$ is a vertex such that $|C \cap \pi| \leq |\pi|/2$ for every connected component $C$ of $G - m$. Then $m$ is a median vertex of profile $\pi$.  

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Proof. Suppose that $x$ is a neighbor of $m$, and let $C$ be the connected component of $T - m$ containing $x$. Then we have

$$d(x, \pi) = \sum_{y \in \pi} d(x, y) = \sum_{y \in C \cap \pi} d(x, y) + \sum_{y \in \pi \setminus C} d(x, y)$$

$$= \sum_{y \in C \cap \pi} (d(m, y) - 1) + \sum_{y \in \pi \setminus C} (d(m, y) + 1)$$

$$= \sum_{y \in \pi} d(m, y) - |C \cap \pi| + |C \cap \pi| \geq \sum_{y \in \pi} d(m, y) = d(m, \pi).$$

Hence the remotness of $x$ with respect to $\pi$ is greater or equal to remotness of $m$ with respect to $\pi$. If $x$ is any other vertex of $C$, the proof follows by induction. □

Corollary 2.6 Let $T$ be a tree of even order. If there exists a vertex $u \in V(T)$ such that no connected component of $T - u$ is larger than $|V(T)|/4$, then $\mu(T) > 0$.

Proof. The strategy of player $I$ is always to take the vertex closest to $u$. Since the size of every connected component of $T - u$ is at most $|V(T)|/4 = |\pi_I|/2 = |\pi_{II}|/2$ we find, by Lemma 2.5, that $u$ is a median of both profiles $\pi_I$ and $\pi_{II}$. If we denote by $x_i$ and $y_i$ the vertex taken by player $I$ and $II$ in $i$-th turn, then we have

$$d(x_i, u) \leq d(y_i, u)$$

for $i \in \{1, \ldots, |V(T)|/2\}$. In particular $x_1 = u$, and therefore $d(x_1, u) < d(y_1, u)$, and hence

$$d(u, \pi_I) < d(u, \pi_{II}).$$

Moreover, $u$ is a median for $\pi_{II}$ by Lemma 2.5, which yields

$$d(u, \pi_{II}) \leq d(v, \pi_{II})$$

for any $v \in \pi_{II}$ and the proof is completed. □

Theorem 2.7 Let $T$ be a tree of even order and suppose that there exists a vertex $m \in V(T)$ such that at most one connected component of $T - m$ has more than $|V(T)|/4$ vertices. If such component has at most $(3|V(T)| - 6)/8$ vertices, then $\mu(T) > 0$.

Proof. Let $|V(T)| = t$ and denote by $L$ the largest component of $T - m$. Note that by Lemma 2.5 $m$ is the median vertex of $T$. The player $I$ has two strategies, A and B, and he uses strategy A until a certain stage of the game (to be declared later) when he switches to strategy B.

- Strategy A: Choose a vertex closest to the median $m$. If there is more than one such vertex, and one of them is in $L$, then choose a vertex closest to the median that is in $L$.  

• Strategy B: If player II has chosen a vertex not in $L$ in his previous turn, then choose a vertex closest to the median $m$. If player II has chosen a vertex in $L$ in his previous turn, then choose a vertex in $L$ that is closest to the median. If such vertex does not exist (that is, all vertices in $L$ are already taken), then choose a vertex closest to the median not in $L$.

Let $s$ denote the number of turns played in the game so far, so we consider the vertices taken by each of the player in the first $s$ turns. Let $x_i$ and $y_i$ be the vertex taken by player I and II in the $i$-th turn, respectively. Let

$a(s)$ be the number of turns when both players have chosen a vertex in $L$.

$b(s)$ be the number of turns when player I has chosen a vertex in $L$ and player II has not.

$c(s)$ be the number of turns when player II has chosen a vertex in $L$ and player I has not.

$d(s)$ be the number of turns when both players have chosen a vertex not in $L$.

The player I uses strategy A until the following condition is true

$$4(c(s) - b(s)) < 2t - 4|L|$$

and when it is false he switches to strategy B. We claim that, at the end of the game, the median of profile $\pi_{II}$ is $m$, and that

$$\sum_{i=1}^{t/2} d(x_i, m) < \sum_{i=1}^{t/2} d(y_i, m).$$

Since $m = x_1 \in \pi_I$ we find that if the above claims are true, player I wins the game. So let us prove both claims. We claim that $|L \cap \pi_{II}| \leq |\pi_{II}|/2$. First note that both sides of (2) are divisible by 4. Let us denote by $s_0$ the smallest $s$ for which (2) is not true, and assume first that such $s_0$ exists. Since the condition (2) is true for $s_0 - 1$ and not for $s_0$, we find that $c(s_0) > c(s_0 - 1)$, and since both sides are divisible by 4 (and $c(s)$ can increase only by 1), we find that

$$4(c(s_0) - b(s_0)) = 2t - 4|L|.$$

According to the definitions of $a, b, c$ and $d$ we see that in the first $s_0$ turns player II gets $a(s_0) + c(s_0)$ vertices in $L$ and in the rest of the game (when player I uses strategy B) player II gets at most

$$\frac{1}{2}(|L| - 2a(s_0) - b(s_0) - c(s_0))$$

vertices in $L$. So all together player II gets at most

$$a(s_0) + c(s_0) + \frac{1}{2}(|L| - 2a(s_0) - b(s_0) - c(s_0)) = \frac{1}{2}(|L| + c(s_0) - b(s_0)) = \frac{t}{4} = \frac{|\pi_{II}|}{2}.$$
vertices of $L$. So we proved that $|L \cap \pi_{II}| \leq |\pi_{II}|/2$ if $s_0$ exists. If $s_0$ does not exist then condition (2) is true for all $s$ except maybe $s = t/2$, where the equality in (2) might be achieved. If we denote $z = t/2$ then we have

$$4(c(z) - b(z)) \leq 2t - 4|L|,$$

and since

$$|L| = 2a(z) + b(z) + c(z),$$

we find that $a(z) + c(z) \leq \frac{1}{4}$ by combining the inequality and the equality above. Since $|L \cap \pi_{II}| = a(z) + c(z)$ we have proved the claim in both cases. Since $|L \cap \pi_{II}| \leq |\pi_{II}|/2$ (and all other components of $T - m$ are not larger than $|\pi_{II}|/2 = t/4$) it follows from Lemma 2.5 that $m$ is a median of profile $\pi_{II}$.

It remains to prove (3). We prove it by arranging vertices of both profiles in pairs, and by comparing the distances of the two vertices in a pair to the median $m$

Now for pairs $(x, y) \in Y$, where both vertices are in $L$, we consider two cases, either $d(x, m) > d(y, m) + 1$ or $d(x, m) \leq d(y, m) + 1$. Assume $x = x_{s+1}$, $y = y_s$ and $d(x, m) > d(y, m) + 1$. Then all vertices on the $y, x$-path are already taken before $(s + 1)$-th turn. Moreover, all such vertices are in $\pi_{II}$ because according to both strategies, $A$ and $B$, the player $I$ would preferred these vertices before $x_{s+1}$. Let $y_p$ be a neighbor of $x$ on the $y, x$-path. Clearly $p < s_0$ by strategies $A$ and $B$. We have $d(x_p, m) < d(y, m)$ by strategy $A$ and $d(y_p, m) + 1 = d(x, m)$. Hence we exchange the pairs $(x_p, y_p) \in X$ and $(x, y) \in Y$ to $(x, y) \in X$ and $(x, y_p) \in Y$ with desired properties $d(x_p, m) < d(y, m)$ and $d(x, m) \leq d(y_p, m) + 1$. Thus we may assume that $d(x, m) \leq d(y, m) + 1$ for all $(x, y) \in Y$ and we get

$$|Y| \leq \frac{1}{2}(|L| - 2a - b - c + 1)$$

which yields
\[
\sum_{(x,y) \in Y} d(y,m) - d(x,m) \geq -|Y| \geq -\frac{1}{2}(|L| - 2a - b - c + 1).
\]

And for pairs \((x,y)\) in \(Z\), we have \(d(x,m) \leq d(y,m)\). Therefore we have
\[
\sum_{(x,y) \in Z} d(y,m) - d(x,m) \geq 0.
\]

Since
\[
\sum_{i=1}^{t/2} d(y_i,m) - \sum_{i=1}^{t/2} d(x_i,m) = 
\sum_{(x,y) \in X} d(y,m) - d(x,m) + \sum_{(x,y) \in Y} d(y,m) - d(x,m) + \sum_{(x,y) \in Z} d(y,m) - d(x,m)
\]
it remains to prove that
\[
(c - 1) - \frac{1}{2}(|L| - 2a - b - c + 1) > 0.
\]

Since \(|L| = t/2 - c + b\) the above inequality is equivalent to
\[
2c + a > \frac{t}{4} + \frac{3}{2}.
\]
But on the other hand we know that \(|L| = t/2 - c + b < (3t - 6)/8\) and hence \(2c - 2b > t/4 + 3/2\), which proves the claim and hence also the theorem. \(\Box\)

The trees discussed in Corollary 2.6 and Theorem 2.7 admit a strategy for player \(I\) to win the game. The strategy is to take always the vertex closest to the median of the tree, or slight alternations of this strategy, where player \(I\) must take care of the largest component of the tree, so that player \(II\) does not get to many vertices of this component. However there are trees for which the described strategy is a losing strategy for player \(I\). Such example is shown in Fig. 1. However, the tree on Fig. 1 has two median vertices: \(x_1\) and \(y_1\) and above described phenomena occur only if player \(I\) starts in \(x_1\). If he would start in \(y_1\), then he wins by this strategy.

References


Figure 1: Graph for which taking vertices close to the median is a losing strategy.


