Bounding the $k$-monopoly number of strong product graphs

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Abstract

Let $G = (V, E)$ be a simple graph without isolated vertices and minimum degree $\delta$, and let $k \in \{1 - \lceil\delta/2\rceil, \ldots, \lfloor\delta/2\rfloor\}$ be an integer. Given a set $M \subset V$, a vertex $v$ of $G$ is said to be $k$-controlled by $M$ if $\delta_M(v) \geq \frac{\delta_G(v)}{2} + k$ where $\delta_M(v)$ represents the quantity of neighbors $v$ has in $M$ and $\delta_G(v)$ the degree of $v$ in $G$. A set $M$ is called a $k$-monopoly if it $k$-controls every vertex $v$ of $G$. The minimum cardinality of any $k$-monopoly is the $k$-monopoly number of $G$. In this article we study the $k$-monopoly number of strong product graphs. We present general lower and upper bounds for the $k$-monopoly number of strong product graphs. Moreover, we study in addition the 0-monopolies of several specific families of strong product graphs.

Keywords: monopolies; strong product graphs; alliances; domination

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1 Introduction and preliminaries

Let $G = (V, E)$ be a simple graph. Given a set $S \subset V$ and a vertex $v \in V$, we denote by $\delta_S(v)$ the number of neighbors $v$ has in $S$. If $S = V$, then $\delta_V(v)$ is the degree of $v$ and we write $\delta_G(v)$ (or just $\delta(v)$, if there are no misunderstandings with the graph $G$). The minimum degree of $G$ is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. If there is no confusion possible, then we use shorter version $\delta$ and $\Delta$ for minimum and maximum degree of $G$. Given an integer $k \in \{1 - \lceil\frac{\delta}{2}\rceil, \ldots, \lfloor\frac{\delta}{2}\rfloor\}$ and a set $M$, the research was partially done while the third author was visiting the University of Maribor, Slovenia, supported by “Ministerio de Educación, Cultura y Deporte”, Spain, under the “Jose Castillejo” program for young researchers. Reference number: CAS12/00267.

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a vertex \( v \) of \( G \) is said to be \( k \)-controlled by \( M \) if \( \delta_M(v) \geq \frac{\delta_G(v)}{2} + k \). The set \( M \) is called a \( k \)-monopoly if it \( k \)-controls every vertex \( v \) of \( G \). The minimum cardinality of any \( k \)-monopoly is the \( k \)-monopoly number and is denoted by \( \mathcal{M}_k(G) \). A monopoly of cardinality \( \mathcal{M}_k(G) \) is called a \( \mathcal{M}_k(G) \)-set. In particular, notice that for a graph with a leaf (vertex of degree one), there exist only 0-monopolies and the neighbor of every leaf is in each \( \mathcal{M}_0 \)-set. Monopolies in graphs were defined first in [12] and they were generalized to \( k \)-monopolies recently in [11]. Other studies about monopolies in graphs and some of its applications can be found in [5, 8, 13, 14, 18].

An equivalent definition for a \( k \)-monopoly in \( G \) can be described in the following way. Let \( \overline{M} \) be the complement of a set \( M \). Hence, a set of vertices \( M \) is a \( k \)-monopoly in \( G \) if and only if for every vertex \( v \) of \( G \), \( \delta_M(v) \geq \delta_{\overline{M}}(v) + 2k \) (from now we will call this expression as the \( k \)-monopoly condition) and we will say that \( M \) is a \( k \)-monopoly in \( G \) if and only if every vertex of \( V(G) \) satisfies the \( k \)-monopoly condition for \( M \).

Several applications to practical problems have been described for monopolies in graphs. The great part of them relating overcoming failures, according to the fact that monopolies have very frequently some common approaches around the notion of majorities: consensus problems [3], diagnosis problems [15] or voting systems [6], among other applications and references. Monopolies in graphs are also closely related to different parameters in graphs. According to several connections which exist between monopolies, global alliances and signed domination in graphs (see [11]), it is known that the complexity of computing the \( k \)-monopoly number of a graph is an NP-hard problem for \( k \geq 0 \), while it is still an open problem for \( k < 0 \). In this sense, it is desirable to study the \( k \)-monopoly number of some particular families of graphs. In this article we obtain general and particular bounds for the \( k \)-monopoly number of strong product graphs.

Studies about graph products have been appearing very frequently in the last few decades and a rich theory involving the structure and recognition of classes of these graphs has emerged, cf. the new book [7]. The most studied graph products are the Cartesian product, the strong product, the direct product, and the lexicographic product, which are also called standard products. A standard approach to graph products is to deduce properties of the product with respect to the same or different properties of its factors. In this sense, the topic which we deal with has been recently studied for the direct product [10]. It is now our goal to study the strong product graph with respect to its monopolies.

From now on, given a graph \( G \), we use the standard notations \( N_G(g) \) for the open neighborhood \( \{g' : gg' \in E(G)\} \) and \( N_G(g) \) for the closed neighborhood \( N_G(g) \cup \{g\} \). The strong product \( G \square H \) of graphs \( G \) and \( H \) is a graph with vertex set \( V(G \square H) = V(G) \times V(H) \). Two vertices \( (g, h) \) and \( (g', h') \) are adjacent in \( G \square H \) whenever \( (gg' \in E(G) \) and \( h = h' \) or \( g = g' \) and \( hh' \in E(H) \)) or \( (gg' \in E(G) \) and \( hh' \in E(H) \)). For a fix \( h \in V(H) \) we call \( G^h = \{(g, h) \in V(G \square H) : g \in V(G)\} \) a \( G \)-layer in \( G \square H \). An \( H \)-layer \( g \) for a fix \( g \in V(G) \) is defined symmetrically. Notice that a subgraph
induced by $G^h$ and $gH$ are isomorphic to $G$ and $H$, respectively. The commutativity of the strong product follows from the symmetry of the definition of adjacency and for associativity, see [7]. The closed neighborhoods of vertices in strong product graphs are nicely connected to closed neighborhoods of projections to the factors:

$$N_{G^h \Box H}[g, h] = N_G[g] \times N_H[h].$$

Notice that the same situation does not occur with open neighborhoods in strong product.

## 2 General bounds

To begin the description of our results we give some general bounds for the $k$-monopoly number of strong product graphs in terms of the order and the monopoly number of the factor graphs. Some of these results are related to the alliances of the factor graphs. Alliances were presented first in [9], and for more information about alliances in graphs we suggest the recent surveys [16, 17].

For $k \in \{-\Delta, \ldots, \Delta\}$, a nonempty set $A \subseteq V$ is a **defensive** $k$-alliance in $G$ if for every $v \in A$ it follows that

$$\delta_A(v) \geq \delta_A^-(v) + k. \quad (1)$$

Moreover, for $k \in \{2 - \Delta, \ldots, \Delta\}$, a nonempty set $A \subseteq V$ is an **offensive** $k$-alliance in $G$ if for every $v \in \partial A$ it follows that

$$\delta_A(v) \geq \delta_A^-(v) + k, \quad (2)$$

where $\partial A$ is the set of all vertices not in $A$ which are adjacent to at least one vertex of $A$.

For $k \in \{-\Delta, \ldots, \Delta - 2\}$, if $A$ is a defensive $k$-alliance as well as an offensive $(k+2)$-alliance, then $A$ is a **powerful** $k$-alliance. A powerful $k$-alliance is called **global** if it is a dominating set. The **global powerful $k$-alliance number** of $G$, denoted by $\gamma_p^k(G)$, is defined as the minimum cardinality of a global powerful $k$-alliance in $G$. For a global powerful $k$-alliance $A$, it is easy to see that $\partial A = \overline{A}$. Hence $\delta_A(v) \geq \delta_A^-(v) + k$ holds for every $v \in A$ and $\delta_A^-(v) \geq \delta_A^-(v) + k + 2$ holds for every $v \in \overline{A}$.

**Theorem 2.1** Let $G$ and $H$ be two graphs without vertices of degree 1 and let $\ell = \min\{\delta(G), \delta(H)\}$. If $k \in \{1 - \lceil \frac{\ell}{2} \rceil, \ldots, \lceil \frac{\ell}{2} \rceil\}$, then

$$\mathcal{M}_k(G \boxtimes H) \leq \min\{\gamma_p^h(G)|V(H)|, |V(G)|\gamma_p^k(H)\}.$$

**Proof.** Let $(g, h)$ be an arbitrary vertex of $G \boxtimes H$ and let $A_G$ and $A_H$ be a $\gamma_p^h(G)$-set and a $\gamma_p^k(H)$-set, respectively. Hence, if $g \in A_G$, then $\delta_{A_G}(g) \geq \delta_{\overline{A_G}}(g) + k$, and if $g \notin A_G$, then $\delta_{A_G}(g) \geq \delta_{\overline{A_G}}(g) + k + 2$. On the other hand, if $h \in A_H$, then $\delta_{A_H}(h) \geq \delta_{\overline{A_H}}(h) + k$,

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and if \( h \notin A_H \), then \( \delta_{A_H}(h) \geq \delta_{\overline{A_H}}(h) + k + 2 \). Hence, we can split the neighborhood \( N_{G \boxtimes H}(g,h) \) of a vertex \((g,h)\) of \( G \boxtimes H \) to the following disjoint union of sets \\
\((A \times C) \cup (B \times C) \cup (A \times D) \cup (B \times D) \cup (\{g\} \times C) \cup (\{g\} \times D) \cup (A \times \{h\}) \cup (B \times \{h\})\),

where \( A = N_C(g) \cap A_G \), \( B = N_C(g) \cap \overline{A_G} \), \( C = N_H(h) \cap A_H \) and \( D = N_H(h) \cap \overline{A_H} \).

Clearly, \(|A \times C| = \delta_{A_G}(g)\delta_{A_H}(h)\), |\( B \times C | = \delta_{\overline{A_G}}(g)\delta_{A_H}(h)\), |\( A \times D | = \delta_{A_G}(g)\delta_{\overline{A_H}}(h)\), |\( B \times D | = \delta_{\overline{A_G}}(g)\delta_{\overline{A_H}}(h)\), |\( \{g\} \times C | = \delta_{A_H}(h)\), |\( \{g\} \times D | = \delta_{\overline{A_H}}(h)\), |\( A \times \{h\} | = \delta_{A_G}(g)\)

and \(|B \times \{h\}| = \delta_{\overline{A_G}}(g)|.

By the commutativity of the strong product, it is enough to show that \( H \) is bounded from above by \(|\{g\} \times \{h\} = \delta_{A_G}(g)\delta_{A_H}(h)\).

Also, \( A \times \{h\} \) and \( B \times \{h\} \) are subsets of \( M \) while \( A \times D, B \times D \) and \( \{g\} \times D \) are subsets of \( \overline{M} \). Sets \( A \times \{h\} \) and \( B \times \{h\} \) are subsets of \( M \) whenever \( h \in A_H \) and subsets of \( \overline{M} \) whenever \( h \in \overline{A_H} \). We will show that \( \delta_M(g,h) \geq \delta_{G \boxtimes H}(g,h) \) for every vertex \((g,h)\) and every \( k \in \{1 - \lceil \ell^2 \rceil, \ldots, \lceil \ell^2 \rceil \} \) where \( \ell = \min\{\delta(G), \delta(H)\} \).

We consider the following cases.

Case 1: If \( h \in A_H \), then, since \( A_H \) is a global defensive \( k \)-alliance, we have that 

\[
\delta_M(g,h) - \delta_{\overline{A_H}}(g,h) = \delta_{A_G}(g)\delta_{A_H}(h) + \delta_{\overline{A_G}}(g)\delta_{A_H}(h) + \delta_{A_G}(g) + \delta_{A_H}(h) + \delta_{\overline{A_G}}(g) + \delta_{\overline{A_H}}(h) + 2k + 1 \geq 2k + 1.
\]

Now, since \( \delta_{G \boxtimes H}(g,h) = \delta_M(g,h) + \delta_{\overline{A_H}}(g,h) \), from the above we have that

\[
\delta_M(g,h) \geq \frac{\delta_{G \boxtimes H}(g,h) + 2k + 1}{2} = \frac{\delta_{G \boxtimes H}(g,h)}{2} + k + \frac{1}{2}.
\]

Case 2: If \( h \in \overline{A_H} \), then since \( A_H \) is a global offensive \((k + 2)\)-alliance, we have

\[
\delta_M(g,h) - \delta_{\overline{A_H}}(g,h) = \delta_{A_G}(g)\delta_{A_H}(h) + \delta_{\overline{A_G}}(g)\delta_{A_H}(h) + \delta_{A_G}(g) + \delta_{A_H}(h) - \delta_{\overline{A_G}}(g) - \delta_{\overline{A_H}}(h) + 2k + 2 \geq 2k + 3.
\]
Similarly to the Case 1, we have that
\[ \delta_M(g, h) \geq \frac{\delta_{G \boxtimes H}(g, h) + 2k + 3}{2} \geq \frac{\delta_{G \boxtimes H}(g, h)}{2} + k + 1. \] (4)

As a consequence of both cases we obtain that \( M \) is a \( k \)-monopoly and the proof is complete. □

If we consider two graphs \( G \) and \( H \) such that all its vertices have even degree, then, since \( k \) is an integer, it follows from the expressions (3) and (4) that every \((g, h)\) of \( V(G \boxtimes H) \) satisfies
\[ \delta_M(g, h) \geq \frac{\delta_{G \boxtimes H}(g, h)}{2} + k + 1 \]
and so, the following result.

**Theorem 2.2** Let \( G \) and \( H \) be two graphs graphs without vertices of degree 1 and let \( \ell = \min\{\delta(G), \delta(H)\} \). If \( G \) and \( H \) are Eulerian graphs, then for \( k \in \{1 - \left\lceil \frac{\ell}{2} \right\rceil, \ldots, \left\lfloor \frac{\ell}{2} \right\rfloor\} \)
\[ \mathcal{M}_{k+1}(G \boxtimes H) \leq \min\{\gamma_k^p(G)|V(H)|, |V(G)|\gamma_k^p(H)\}. \]

To give some lower bounds for the \( k \)-monopoly number of strong product graphs we need some additional terminology. Given a graph \( G \) and a set \( S \subset V(G) \), we say that closed neighborhoods over \( S \) form a closed subpartition for \( G \) if \( N_G[u] \cap N_G[v] = \emptyset \) for every different \( u, v \in S \). A closed subpartition \( S \) for \( G \) is a maximum closed subpartition, if \( V(G) - \cup_{v \in S} N_G[v] \) has the minimum cardinality among all closed subpartitions \( S' \) for \( G \). Notice that, if \( S \) forms a maximum closed subpartition which is also a partition of \( V(G) \), then \( S \) is called a perfect code [2] or an efficient dominating set [1] of \( G \). We have the following result, which yields a corollary important for this work.

**Proposition 2.3** Let \( G \) and \( H \) be graphs. If \( S_G \) and \( S_H \) form maximum closed subpartitions for \( G \) and \( H \), respectively, then \( S = S_G \times S_H \) form a closed subpartition for \( G \boxtimes H \).

**Proof.** Let \( S_G \) and \( S_H \) be maximum closed subpartitions for \( G \) and \( H \), respectively, and let \( S = S_G \times S_H \). Since \( N_{G \boxtimes H}[(g, h)] = N_G[g] \times N_H[h] \) holds, we have \( N_{G \boxtimes H}[g, h] \cap N_{G \boxtimes H}[g', h'] = \emptyset \) for every different vertices \((g, h)\) and \((g', h')\) from \( S \). □

Now we present lower bounds on the monopoly number of strong product graphs. One of them will be based on the following observation.

**Observation 2.4** Let \( G \) be a graph. If \( S \subset V(G) \) forms a closed subpartition of \( G \), then
\[ \mathcal{M}_k(G) \geq k|S| + \sum_{v \in S} \left\lceil \frac{\delta(v)}{2} \right\rceil. \]
The following theorem follows directly from the above observation, Proposition 2.3 and the fact that
\[ \delta_{G\boxtimes H}(g, h) = \delta_G(g)\delta_H(h) + \delta(g) + \delta(h). \]

**Theorem 2.5** Let \( G \) and \( H \) be graphs without isolated vertices and let \( \ell = \delta(G)\delta(H) + \delta(G) + \delta(H) \). If \( S_G \) and \( S_H \) are maximum closed subpartitions of \( G \) and \( H \), respectively, then for \( k \in \{1 - \left\lceil \frac{\ell}{2} \right\rceil, \ldots, \left\lfloor \frac{\ell}{2} \right\rfloor \} \) we have
\[
M_k(G \boxtimes H) \geq k|S_G||S_H| + \sum_{(g, h) \in S_G \times S_H} \left\lceil \frac{\delta_G(g)\delta_H(h) + \delta(g) + \delta(h)}{2} \right\rceil.
\]

The next result, presented previously in [10], is also useful to obtain a lower bound on the monopoly number of strong product graphs.

**Proposition 2.6** [10] Let \( G \) be a graph of order \( n \), minimum degree \( \delta \) and maximum degree \( \Delta \). Then, for any \( k \in \{1 - \left\lceil \frac{\delta}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta}{2} \right\rfloor \} \)
\[
M_k(G) \geq \left\lceil \frac{n}{\Delta} \left( \left\lfloor \frac{\delta}{2} \right\rfloor + k \right) \right\rceil.
\]

**Corollary 2.7** Let \( G \) and \( H \) be two graphs without isolated vertices of order \( n \) and \( m \), respectively. For any \( k \in \{1 - \left\lfloor \frac{\delta(G)\delta(H)}{2} \right\rceil, \ldots, \left\lceil \frac{\delta(G)\delta(H)}{2} \right\rfloor \} \) we have
\[
M_k(G \boxtimes H) \geq \left\lceil \frac{mn}{\Delta(G)\Delta(H) + \Delta(G) + \Delta(H)} \left( \left\lceil \frac{\delta(G)\delta(H) + \delta(G) + \delta(H)}{2} \right\rceil + k \right) \right\rceil.
\]

The best performance of the last bound is achieved when both factors are Eulerian regular graphs.

### 3 Bounds or closed formulae for particular families of strong product graphs

We start with the strong product of two cycles.

**Theorem 3.1** Let \( r, t \geq 3 \), \( r \leq t \), be two integers. If \( r \) and \( t \) are even, then \( M_0(C_r \boxtimes C_t) = \frac{rt}{2} \). Moreover, if at least one of integers \( r \) or \( t \), say \( t \), is odd, then
\[
\left\lceil \frac{rt}{2} \right\rceil \leq M_0(C_r \boxtimes C_t) \leq \left\lceil \frac{rt}{2} \right\rceil + \left\lceil \frac{r}{4} \right\rceil, \quad \text{if } r \text{ is even,}
\]
\[
\left\lfloor \frac{rt}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor + \left\lceil \frac{t}{4} \right\rceil, \quad \text{if } r \text{ is odd.}
\]
Proof. Let $V(C_r) = \{u_0, \ldots, u_{r-1}\}$ and $V(C_t) = \{v_0, \ldots, v_{t-1}\}$. From now on, operations with subscripts of vertices of $C_r$ and $C_t$ are done modulo $r$ and modulo $t$, respectively. Also we assume that $u_i, u_{i+1}$ and $v_i, v_{i+1}$ are adjacent in $C_r$ and $C_t$, respectively. Clearly, $C_r \boxtimes C_t$ is an 8-regular graph and $\mathcal{M}_0(C_r \boxtimes C_t) \geq \left\lfloor \frac{rt}{2} \right\rfloor$ holds by Corollary 2.7.

Let $M = \{(u_i, v_j) : i, j \text{ have different parities}\}$. Notice that the cardinality of $M$ is $\left\lfloor \frac{rt}{2} \right\rfloor$. We suppose first that both $r$ and $t$ are even. Hence, it is easy to check that every vertex $(u_i, v_j) \in V(C_r \boxtimes C_t)$ has exactly four neighbors in $M$ and four neighbors in $\overline{M}$. So, $M$ is a 0-monopoly in $C_r \boxtimes C_t$. Moreover, in every $C_r$-layer there are exactly $\frac{r}{2}$ vertices of $M$, which gives all together $\frac{rt}{2}$ vertices in $M$. Therefore, we have that $\mathcal{M}_0(C_r \boxtimes C_t) = \frac{rt}{2}$.

Let now $t \geq 3$ be an odd integer. We consider first $r \geq 3$ being an even integer. It is straightforward to observe that all the vertices of $M$ belonging to the set $V(C_r \boxtimes C_t) \setminus (C_r^{u_0} \setminus C_t^{v_1})$ are 0-controlled by $M$ (we recall that $C_r^{u_0}$ is the $C_r$-layer corresponding to the vertex $v_0$). Vertices from $C_r^{u_0} \cup C_t^{v_1}$ which are not in $M$ have five neighbors in $M$ and are also 0-controlled. On the other hand, vertices of $M$ in $C_r^{u_0} \cup C_t^{v_1}$ have only three neighbors in $M$. Hence, it is necessary to add some extra vertices to $M$ to have a 0-monopoly in $C_r \boxtimes C_t$. That is, $M_1 = M \cup \{(u_{4i}, v_{t-1}) : i \in \{0, \ldots, \left\lfloor \frac{r}{4} \right\rfloor - 1\}\}$ is a 0-monopoly in $C_r \boxtimes C_t$ and, thus, $\mathcal{M}_0(C_r \boxtimes C_t) \leq \frac{rt}{2} + \left\lfloor \frac{r}{4} \right\rfloor$.

If $r \geq 3$ is an odd integer, then we make an analogue extension of $M$ to a set $M_2$ as we did above from $M$ to $M_1$. That is,

$M_2 = M \cup \{(u_{4i}, v_{t-1}) : i \in \{0, \ldots, \left\lfloor \frac{r}{4} \right\rfloor - 1\}\} \cup \{(u_{r-1}, v_{4j+2}) : j \in \{0, \ldots, \left\lfloor \frac{t}{4} \right\rfloor - 1\}\}$.

Similarly as before, $M_2$ is a 0-monopoly in $C_r \boxtimes C_t$. Thus, it follows

$$\mathcal{M}_0(C_r \boxtimes C_t) \leq \left\lfloor \frac{rt}{2} \right\rfloor + \left\lfloor \frac{r}{4} \right\rfloor + \left\lfloor \frac{t}{4} \right\rfloor.$$ 

By using a similar pattern like in the above result, we obtain the following for the case of strong product of paths.

Proposition 3.2 If $r \geq 3$ and $t \geq 3$ are two integers, then

$$\mathcal{M}_0(P_r \boxtimes P_t) \leq \left\lfloor \frac{rt}{2} \right\rfloor + \left\lfloor \frac{r + t - 2}{2} \right\rfloor.$$ 

Proof. Let $V(P_r) = \{u_1, \ldots, u_r\}$ and $V(P_t) = \{v_1, \ldots, v_t\}$. With the above notation we suppose that two consecutive vertices of $V(P_i)$ are adjacent for $i \in \{r, t\}$. Suppose that $r, t$ have the same parity. Consider the set $M = \{(u_i, v_j) :
\(i, j\) have different parities). Notice that the cardinality of \(M\) is \(\left\lceil \frac{rt}{2} \right\rceil\). First we consider the case that \(r\) and \(t\) are odd integers. Now, it is easy to see that any vertex \((u_i, v_j) \in V(P_r \otimes P_t)\) with \(\delta_{P_r \otimes P_t}(u_i, v_j) = 8\) has exactly four neighbors in \(M\) and four neighbors in \(\overline{M}\). Also, if \(\delta_{P_r \otimes P_t}(u_i, v_j) = 3\), then \(\delta_M(u_i, v_j) = 2\), and if \(\delta_{P_r \otimes P_t}(u_i, v_j) = 5\) and \((u_i, v_j) \notin M\), then \(\delta_M(u_i, v_j) = 3\). However, if \(\delta_{P_r \otimes P_t}(u_i, v_j) = 5\) and \((u_i, v_j) \in M\), then \(\delta_M(u_i, v_j) = 2\) which is not enough to satisfy the 0-monopoly condition, and we need to add some additional vertices to \(M\). Notice that there are exactly \(2^{r-1+t-1} = r + t - 2\) of such vertices, and this is an even number, since \(r, t\) have the same parity. Since, the layers \(P_r^1, P_t^1, P_r^r\) and \(1P_t\) form a cycle, say \(C\), and any two consecutive vertices of \(M\) lying in \(C\) have a common neighbor in \(C\) not belonging to \(M\), we can split them into pairs and for each pair we add to \(M\) one extra vertex, to obtain a 0-monopoly set in \(P_r \otimes P_t\). Therefore we have

\[
\mathcal{M}_0(P_r \otimes P_t) \leq \left\lceil \frac{r t}{2} \right\rceil + \frac{r + t - 2}{2}.
\]

Now, we consider that \(r, t\) are even integers. We proceed similarly as above, but in this case, there also exist two vertices with \(\delta_{P_r \otimes P_t}(u_i, v_j) = 3\), for which \(\delta_M(u_i, v_j) = 1\) (not satisfying the 0-monopoly condition). Again, if \(\delta_{P_r \otimes P_t}(u_i, v_j) = 5\) and \((u_i, v_j) \in M\), then \(\delta_M(u_i, v_j) = 2\) which is not enough to satisfy the 0-monopoly condition, and we need to add some additional vertices to \(M\). There are exactly \(2^{r-1+t-1} = r + t - 2\) of such vertices (two with degree three and the rest with degree five). The same lower bound is obtained as above.

On the other hand, without loss of generality, suppose \(t \geq 3\) is even and \(r \geq 3\) is odd. We consider the same set \(M\), which has the same cardinality as above. Also, only those vertices \((u_i, v_j) \in M\) with \(\delta_{P_r \otimes P_t}(u_i, v_j) = 5\) and two corner vertices of degree three are not satisfying the 0-monopoly condition, since \(\delta_M(u_i, v_j) = 2\) or \(\delta_M(u_i, v_j) = 1\), respectively. Thus, we need to add some additional vertices to \(M\). Again, there are exactly \(2^{r-1+t-1} = r + t - 2\) of such a vertices, but in this case, this is an odd number, since \(r, t\) have different parities. By adding \(\left\lceil \frac{r + t - 2}{2} \right\rceil\) vertices to \(M\), in a similar way, we obtain a 0-monopoly. Thus

\[
\mathcal{M}_0(P_r \otimes P_t) \leq \left\lceil \frac{r t}{2} \right\rceil + \left\lceil \frac{r + t - 2}{2} \right\rceil.
\]

\(\square\)

We have a somewhat combination of the above two results when one factor is a path and the second one is a cycle. Since the construction of a 0-monopoly set and the proof are very similar to the above ones, we omit it and leave it to the reader.
Proposition 3.3 If \( r \geq 3 \) and \( t \geq 3 \) are two integers, then

\[
\mathcal{M}_0(P_r \boxtimes C_t) \leq \begin{cases} 
\frac{rt}{2} + 2 \left\lceil \frac{t}{4} \right\rceil, & \text{if } t \text{ is even}, \\
\frac{rt}{2} + \left\lceil \frac{r}{4} \right\rceil + 2 \left\lfloor \frac{t}{4} \right\rfloor, & \text{if } r \text{ is even and } t \text{ is odd}, \\
\frac{rt-1}{2} + \left\lceil \frac{r}{4} \right\rceil + 2 \left\lfloor \frac{t}{4} \right\rfloor, & \text{if } r, t \text{ are odd}.
\end{cases}
\]

The upper bound of \( k \)-monopolies of strong product graphs depends on the value of the global powerful \( k \)-alliance number of its factors by Theorem 2.1. For the complete graphs it is as follows.

Lemma 3.4 [4] For any \( K_t \) of order \( t \geq 2 \) and \( k \in \{2 - t, \ldots, t - 3\} \),

\[
\gamma_p^k(K_t) = \left\lceil \frac{t + k + 1}{2} \right\rceil.
\]

Since \( C_r \boxtimes K_t \) is a regular graph of degree \( 3t - 1 \), from Theorem 2.1, Corollary 2.7, Lemma 3.4, and by making some straightforward calculations, it is possible to observe that for any \( r, t \geq 3 \),

\[
\left\lceil \frac{rt}{2} \right\rceil \leq \mathcal{M}_0(C_r \boxtimes K_t) \leq r \left\lceil \frac{t + 1}{2} \right\rceil.
\]

Nevertheless such results can be improved as we show at next.

Proposition 3.5 For integers \( r, t \geq 3 \) we have

\[
\frac{rt}{2} + \frac{r}{6} \leq \mathcal{M}_0(C_r \boxtimes K_t) \leq r \left\lceil \frac{t + 1}{2} \right\rceil - \left\lfloor \frac{r}{3} \right\rfloor.
\]

Moreover, if \( r \equiv 0 \pmod{3} \) and \( t \) is odd, then \( \mathcal{M}_0(C_r \boxtimes K_t) = \frac{rt}{2} + \frac{r}{6} \).

Proof. Let \( V(C_r) = \{u_0, \ldots, u_{r-1}\} \) and \( V(K_t) = \{v_1, \ldots, v_t\} \). With the above notation we suppose that two consecutive vertices \( u_i \) and \( u_{i+1} \) are adjacent (all the operations with the subindexes of \( u_i \) are done modulo \( r \)). Let \( M \) be a \( \mathcal{M}_0(C_r \boxtimes K_t) \)-set and let \( (u_i, v_j) \in M \). Hence, we have that

\[
\delta_M(u_i, v_j) \geq \frac{\delta_{C_r \boxtimes K_t}(u_i, v_j)}{2} = \frac{3t - 1}{2}.
\]

Now, for every \( i \in \{0, \ldots, r - 1\} \), let \( M_i = M \cap (\{u_{i-1}, u_i, u_{i+1}\} \times V(K_t)) \). Hence, for every \( (x, y) \in M_i \), from (5) we obtain that \( |M_i| \geq \delta_{M_i}(x, y) \geq \frac{3t - 1}{2} \). On the other hand, it is clear that for every \( i \in \{0, \ldots, r - 1\} \), \( M_i \neq \emptyset \). Also, notice that if there exist \( j \in \{0, \ldots, r - 1\} \) such that \( M \cap (\{u_j\} \times V(K_t)) = \emptyset \), then \( M \cap (\{u_l\} \times V(K_t)) \neq \emptyset \) for every \( l \in \{j - 2, j - 1, j + 1, j + 2\} \). Thus, there exist at least \( \frac{r}{3} \) disjoint sets \( M_i \).
satisfying $|M_i| \geq \frac{3r-1}{2} + 1$, since we need to count one vertex of $M_i$ and its neighbors inside of the set $M_i$. Therefore, we obtain that

$$|M| \geq \frac{1}{3} \left( \sum_{i=0}^{r-1} |M_i| \right) \geq \frac{1}{3} \left( \sum_{i=0}^{r-1} \left( \frac{3t-1}{2} \right) \right) + \frac{r}{3} = \frac{r(3t-1)}{6} + \frac{r}{3} = \frac{rt}{2} + \frac{r}{6},$$

and the lower bound is proved.

Let $S_{C_r}$ be a maximum closed subpartition for $C_r$. Let $A$ be a set such that if $u_i \in S_{C_r}$, then $\left\lfloor \frac{r}{3} \right\rfloor$ vertices of the $uK_t$-layer belong to $A$, otherwise $\left\lceil \frac{r}{3} \right\rceil$ vertices of the $uK_t$-layer belong to $A$. We will prove that $A$ is a 0-monopoly in $C_r \boxtimes K_t$. Since $C_r \boxtimes K_t$ is a $(3t-1)$-regular graph, we only need to show that any vertex $(u_i, v_j) \in V(C_r \boxtimes K_t)$ has at least $\left\lceil \frac{3r-1}{2} \right\rceil$ neighbors in $A$. If $(u_i, v_j) \notin A$, then $|N_{C_r \boxtimes K_t}(u_i, v_j) \cap A| \geq 3 \left\lceil \frac{r}{2} \right\rceil + 2 > \left\lceil \frac{3r-1}{2} \right\rceil$. Otherwise, if $(u_i, v_j) \in A$, then $|N_{C_r \boxtimes K_t}(u_i, v_j) \cap A| \geq 3 \left\lceil \frac{r}{2} \right\rceil + 1 \geq \left\lceil \frac{3r-1}{2} \right\rceil$. Thus, the 0-monopoly condition holds. Notice that $|A| = r \left( \left\lceil \frac{r}{2} \right\rceil + 1 \right) - |S_{C_r}| = r \left\lceil \frac{r+1}{2} \right\rceil - \left\lceil \frac{r}{3} \right\rceil$, which completes the proof of the upper bound.

Now, if $r \equiv 0 (\text{mod } 3)$ and $t$ is odd, then $|S_{C_r}| = \frac{r}{3}$ and we have that

$$\mathcal{M}_0(C_r \boxtimes K_t) \leq r \left\lceil \frac{t+1}{2} \right\rceil - |S_{C_r}| = \frac{r(t+1)}{2} - \frac{r}{3} = \frac{rt}{2} + \frac{r}{6}.$$

Therefore the equality follows for this case. \hfill \square

**Proposition 3.6** For $r, t \geq 3$ we have

$$\frac{tr}{2} + \frac{r}{6} - \frac{t}{3} \leq \mathcal{M}_0(P_r \boxtimes K_t) \leq \begin{cases} \left\lfloor \frac{r}{3} \right\rfloor \cdot \left\lceil \frac{3r+1}{2} \right\rceil, & \text{if } r \equiv 0 \pmod{3}, \\
\left\lceil \frac{r}{3} \right\rceil \cdot \left\lceil \frac{3r+1}{2} \right\rceil + 1, & \text{if } r \equiv 1 \pmod{3}, \\
\left\lfloor \frac{r}{3} \right\rfloor \cdot \left\lceil \frac{3r+1}{2} \right\rceil + t, & \text{if } r \equiv 2 \pmod{3}. \end{cases}$$

**Proof.** Let $V(P_r) = \{u_1, \ldots, u_r\}$ and $V(K_t) = \{v_1, \ldots, v_t\}$. With the above notation we suppose that two consecutive vertices of $V(P_r)$ are adjacent. Let $M$ be a $\mathcal{M}_0(P_r \boxtimes K_t)$-set and let $(u_i, v_j) \in M \cap (V(P_r) - \{u_1, u_r\}) \times V(K_t)$. Hence we have that

$$\delta_M(u_i, v_j) \geq \frac{\delta_{P_r \boxtimes K_t}(u_i, v_j)}{2} = \frac{3t-1}{2}. \quad (6)$$

Now, for every $i \in \{0, \ldots, r+1\}$, let $M_i = M \cap \{(u_{i-1}, u_i, u_{i+1}) \times V(K_t)\}$ where $u_{-1}, u_0, u_{r+1}, u_{r+2}$ are imaginary vertices needed for counting three times every vertex from $M$. It is clear that for every $i \in \{1, \ldots, r\}$, $M_i \neq \emptyset$. Hence, for every $(x, y) \in M_i$, from [6] we obtain that $|M_i| \geq \delta_{M_i}(x, y) \geq \frac{3r-1}{2}$. Now, analogously to the proof of Proposition 3.5, there exist at least $\frac{r}{3}$ disjoint sets $M_i$ satisfying that $|M_i| \geq \frac{3r-1}{2} + 1$. Moreover, we
need at least $\frac{2t-1}{2}$ vertices in $M_1$ and at least $\frac{2t-1}{2}$ in $M_r$, since $\delta_{P_r \boxtimes K_t}(u, v) = 2t - 1$ for $i \in \{1, r\}$. Thus, we obtain that

$$|M| \geq \frac{1}{3} \sum_{i=0}^{r+1} |M_i|$$

$$\geq \frac{1}{3} \left( |M_0| + |M_1| + |M_r| + |M_{r+1}| + \sum_{i=2}^{r-1} \frac{3t-1}{2} \right) + \frac{r}{3}$$

$$\geq \frac{2t-1}{3} + \frac{(r-2)(3t-1)}{6} + \frac{r}{3}$$

$$= \frac{2r}{6} + r - \frac{t}{3},$$

and the lower bound is proved. To prove the upper bound we consider the following cases.

Case 1: $r \equiv 0 \pmod{3}$. Let $A_0$ be a set such that if $i \equiv 2 \pmod{3}$, then the whole $uK_t$-layer is a subset of $A_0$ and, for every $i \equiv 0 \pmod{3}$, $\lceil \frac{t+1}{2} \rceil$ vertices of the $uK_t$-layer belong to $A_0$. We show that $A_0$ is a 0-monopoly set in $P_r \boxtimes K_t$. Let $(u, v)$ be any vertex in $P_r \boxtimes K_t$. We differentiate the following cases.

Case 1.1: $i \equiv 0 \pmod{3}$. Then, $\delta_{P_r \boxtimes K_t}(u, v) = 2t-1$ and $\delta_{P_r \boxtimes K_t}(u, v) = 3t-1$ for $i \neq r$. From the construction of the set $A_0$, we have that $\delta_{A_0}(u, v) = t + \lceil \frac{t+1}{2} \rceil = \lceil \frac{3t+1}{2} \rceil$ if $(u, v) \notin A_0$, and $\delta_{A_0}(u, v) = t + \lceil \frac{t+1}{2} \rceil - 1 = \lceil \frac{3t-1}{2} \rceil$ if $(u, v) \in A_0$. Thus, the 0-monopoly condition holds.

Case 1.2: $i \equiv 1 \pmod{3}$. Hence, $\delta_{P_r \boxtimes K_t}(u, v) = 2t-1$ and $\delta_{P_r \boxtimes K_t}(u, v) = 3t-1$ for $i \neq 1$. From the construction of the set $A_0$, we have that $\delta_{A_0}(u, v) = t$ and $\delta_{A_0}(u, v) = t + \lceil \frac{t+1}{2} \rceil = \lceil \frac{3t+1}{2} \rceil$, when $i \neq 1$. So, the 0-monopoly condition again holds.

Case 1.3: $i \equiv 2 \pmod{3}$. From the construction of the set $A_0$ we obtain that $\delta_{A_0}(u, v) = t - 1 + \lceil \frac{t+1}{2} \rceil = \lceil \frac{3t-1}{2} \rceil$. Since $\delta_{P_r \boxtimes K_t}(u, v) = 3t - 1$, the 0-monopoly condition holds.

Notice that $|A_0| = \lceil \frac{r}{3} \rceil \left( t + \lceil \frac{t+1}{2} \rceil \right) = \lceil \frac{r}{3} \rceil \cdot \lceil \frac{3t+1}{2} \rceil$, which completes the proof of Case 1.

Case 2: $r \equiv 1 \pmod{3}$. Let $A_1$ be a set such that if $i \equiv 2 \pmod{3}$ and $i < r - 2$, then $uK_t \subset A_1$ and $u^{r-1}K_t \subset A_1$. Moreover, for every $i \equiv 0 \pmod{3}$ such that $i < r - 1$, $\lceil \frac{t+1}{2} \rceil$ vertices of the $uK_t$-layer belong to $A_1$ and $\lceil \frac{t+3}{2} \rceil$ vertices of the $u^{r-2}K_t$-layer belong to $A_1$. By using an analogous procedure, like in Case 1, we can show that $A_1$ is a 0-monopoly set in $P_r \boxtimes K_t$. Notice that $|A_1| = |A_0| + 1$ and the proof of Case 2 is complete.

Case 3: $r \equiv 2 \pmod{3}$. Let $A_2$ be a set such that if $i \equiv 2 \pmod{3}$ and $i < r - 2$, then $uK_t \subset A_2$ and $u^{r-1}K_t \subset A_2$. Moreover, for every $i \equiv 0 \pmod{3}$, $\lceil \frac{t+1}{2} \rceil$ vertices
of the "$K_t$-layer belong to $A_2$. An analogous procedure, like in Case 1, can be used to show that $A_2$ is a 0-monopoly set in $P_t \boxtimes K_t$. Notice that $|A_2| = |A_0| + t$, which completes the proof. □

References


