On the simultaneous metric dimension of a graph and its complement

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Abstract

A set $S \subseteq V(G)$ is a resolving set of a graph $G$ if, for any two distinct vertices $x$ and $y$ of $G$, there exists a vertex $z \in S$ such that $d(x, z) \neq d(y, z)$, where $d(u, v)$ denotes the length of a shortest path between vertices $u$ and $v$ in $G$. The metric dimension $\dim(G)$ of $G$ is the minimum of the cardinalities of all resolving sets of $G$. Ramírez-Cruz, Oellermann and Rodríguez-Velázquez [Discrete Appl. Math. 198 (2016) 241-250] introduced the notion of a simultaneous resolving set and the simultaneous metric dimension of a graph family. A set $S \subseteq V$ is a simultaneous resolving set for a finite collection $C$ of graphs on a common vertex set $V$ if $S$ is a resolving set for every graph in $C$; the minimum among the cardinalities of all such $S$ is called the simultaneous metric dimension of $C$, denoted by $\text{Sd}(C)$. In this paper, we focus on the simultaneous metric dimension of a graph and its complement. We characterize graphs $G$ satisfying $\text{Sd}(G, \overline{G}) = 1$ and $\text{Sd}(G, \overline{G}) = \card{V(G)} - 1$, respectively, where $\overline{G}$ denotes the complement of $G$. We show that $\{\text{diam}(G), \text{diam}(\overline{G})\} \neq \{3\}$ implies $\text{Sd}(G, \overline{G}) = \max\{\dim(G), \dim(\overline{G})\}$. We construct a family of self-complementary split graphs $G$ of diameter 3 satisfying $\text{Sd}(G, \overline{G}) > \max\{\dim(G), \dim(\overline{G})\}$. We determine $\text{Sd}(G, \overline{G})$ when $G$ is a tree or a unicyclic graph. We conclude the paper with some open problems.

Keywords: diameter, metric dimension, simultaneous resolving set, simultaneous metric dimension, complement of a graph, trees, unicyclic graphs

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1 Introduction

Let $G$ be a finite, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices $x, y \in V(G)$, denoted by $d_G(x, y)$, is the minimum number of edges on a path connecting $x$ and $y$ in $G$, should such a path exist; otherwise, we regard the distance as infinity and write $d_G(x, y) = \infty$. The diameter $\text{diam}(G)$ of $G$ is $\max\{d_G(x, y) : x, y \in V(G)\}$. The open neighborhood of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree $\deg_G(u)$ of a vertex $u$ in $G$ is $\card{N_G(u)}$, the cardinality of $N_G(u)$. An end vertex is a vertex of degree one; a support vertex is a vertex that is adjacent to an end vertex, and a major vertex is a vertex of degree at least three. The complement of $G$, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G})$ such that $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$ for any distinct $x, y \in V(G)$. We denote
by \(P_n, C_n, K_n, K_{n-n-a}\), respectively, the path, the cycle, the complete graph, and the complete bipartite graph on \(n\) vertices. Let \([k] = \{1, 2, \ldots, k\}\) and \([k]_0 = [k] \cup \{0\}\).

A vertex \(z \in V(G)\) resolves a pair of vertices \(x, y \in V(G)\) if \(d_G(x, z) \neq d_G(y, z)\). For two distinct vertices \(x, y \in V(G)\), let \(R_G(x, y) = \{z \in V(G) : d_G(x, z) \neq d_G(y, z)\}\). A set \(S \subseteq V(G)\) is a resolving set of \(G\) if \(|S \cap R_G(x, y)| \geq 1\) for every pair of distinct vertices \(x\) and \(y\) of \(G\). The metric dimension \(\dim(G)\) of \(G\) is the minimum of the cardinalities of all resolving sets of \(G\), and a resolving set of \(G\) having cardinality \(\dim(G)\) is called a (metric) basis of \(G\). For an ordered set \(S = \{u_1, \ldots, u_k\} \subseteq V(G)\) of distinct vertices, the (metric) code of \(v \in V(G)\) with respect to \(S\) in \(G\) is the \(k\)-vector \(c_G(S, v) = (d_G(v, u_1), \ldots, d_G(v, u_k))\). Introduced by Slater [16] and by Harary and Melter [7], the metric dimension of graphs has been extensively studied. It is noted in [6] that determining the metric dimension of a graph is an NP-hard problem. One of the motivations for studying metric dimension is robot navigation (see [10]), where a robot determines its location in the network by landmarks or transmitters placed at nodes (vertices of the graph) in the network modeled by a graph; thus, metric dimension is the minimum number of landmarks or transmitters required for the robot to know its location at all nodes in the network.

In 1990, Brigham and Dutton [3] considered a family of spanning subgraphs \(\{H_1, \ldots, H_t\}\), called factors, of a given graph; they studied the domination parameter for partitions \(\{G, \overline{G}\}\) of \(K_n\). In 2016, for a given family \(\mathcal{C} = \{H_1, \ldots, H_t\}\) of connected graphs \(H_i = (V, E_i)\) with common vertex set \(V\), Ramírez-Cruz et al. [15] defined a simultaneous resolving set for \(\mathcal{C}\) to be a set \(S \subseteq V\) such that \(S\) is a resolving set for every graph \(H_i\) for \(i \in [t]\), where the edge sets of the graphs from \(\mathcal{C}\) are not necessarily disjoint. The simultaneous metric dimension of \(\mathcal{C}\), denoted by \(\text{Sd}(\mathcal{C})\) or \(\text{Sd}(H_1, \ldots, H_t)\), is the minimum among the cardinalities of all simultaneous resolving sets of \(\mathcal{C}\). For other articles on simultaneous metric dimension or its variations, see [1, 5, 12, 13, 14].

In this paper, we investigate the simultaneous metric dimension for a pair of subgraphs \(\{G, \overline{G}\}\) of \(K_n\), where \(n \geq 2\). The results in the next section will show that the study of \(\text{Sd}(G, \overline{G})\) naturally splits into three cases, according to \((\text{diam}(G), \text{diam}(\overline{G}))\): (i) \((\text{diam}(G), \text{diam}(\overline{G})) = (1, \infty)\); (ii) \((\text{diam}(G), \text{diam}(\overline{G})) = (2, k)\), where \(k \geq 2\); (iii) \((\text{diam}(G), \text{diam}(\overline{G})) = (3, 3)\). Case (i) is trivial, and any resolving set of \(G\) is a resolving set of \(\overline{G}\) in case (ii), leaving case (iii) to closer scrutiny.

The paper is organized as follows. In Section 2, we obtain some general results on \(\text{Sd}(G, \overline{G})\). Noting that \(1 \leq \max\{\text{dim}(G), \text{dim}(\overline{G})\} \leq \text{Sd}(G, \overline{G}) \leq \min\{\text{dim}(G) + \text{dim}(\overline{G}), |V(G)| - 1\} \leq |V(G)| - 1\) for any non-trivial graph \(G\), we characterize graphs \(G\) satisfying \(\text{Sd}(G, \overline{G}) = 1\) and \(\text{Sd}(G, \overline{G}) = |V(G)| - 1\), respectively. We also show that, if \(\{\text{diam}(G), \text{diam}(\overline{G})\} \neq \{3\}\), then \(\text{Sd}(G, \overline{G}) = \max\{\text{dim}(G), \text{dim}(\overline{G})\}\). In Section 3, we consider graphs \(G\) with \(\text{diam}(G) = 3 = \text{diam}(\overline{G})\). We determine \(\text{Sd}(G, \overline{G})\) when \(G\) is a tree or a unicyclic graph with \(\text{diam}(G) = 3 = \text{diam}(\overline{G})\). We construct a family of self-complementary split graphs \(G\) of diameter 3 with \(\text{Sd}(G, \overline{G}) > \max\{\text{dim}(G), \text{dim}(\overline{G})\}\).

In Section 4, we examine \(\text{Sd}(G, \overline{G})\) when \(G\) is a tree or a unicyclic graph. For any non-trivial tree \(T \neq P_4\), we show that \(\text{Sd}(T, \overline{T}) = \text{dim}(T) \geq \text{dim}(T)\) and characterize tree \(T\) satisfying \(\text{Sd}(T, \overline{T}) = \text{dim}(T)\). We also show that, for any tree \(T\) that is not a path, \(\text{dim}(\overline{T}) \geq \text{dim}(T) + |\text{end} (T) - 1|\), where \(\text{end}(T)\) denotes the number of end vertices of \(T\). For any unicyclic graph \(G\) of order \(n \geq 3\), we show that \(\text{Sd}(G, \overline{G}) \in \{\text{dim}(G) + 1 + \text{dim}(\overline{G})\}\); moreover, we show that \(\text{Sd}(G, \overline{G}) = \text{dim}(\overline{G})\) for \(n \geq 7\).

In Section 5, we state some interesting observations and conclude with some open problems.

## 2 General results on \(\text{Sd}(G, \overline{G})\)

In this section, we obtain some general results on \(\text{Sd}(G, \overline{G})\). A result from [15] implies that, for any non-trivial graph \(G\), \(1 \leq \max\{\text{dim}(G), \text{dim}(\overline{G})\} \leq \text{Sd}(G, \overline{G}) \leq \min\{\text{dim}(G) + \text{dim}(\overline{G}), |V(G)| - 1\} \leq |V(G)| - 1\). We characterize graphs \(G\) satisfying \(\text{Sd}(G, \overline{G}) = 1\) and \(\text{Sd}(G, \overline{G}) = |V(G)| - 1\), respectively. We also show that, if \(\{\text{diam}(G), \text{diam}(\overline{G})\} \neq \{3\}\), then \(\text{Sd}(G, \overline{G}) = \max\{\text{dim}(G), \text{dim}(\overline{G})\}\).
We begin with some useful observations. Two vertices \( u, w \in V(G) \) are called twin vertices if \( N_G(u) - \{ w \} = N_G(w) - \{ u \} \).

**Observation 2.1.** [8] Two distinct vertices \( u \) and \( w \) are twin vertices in \( G \) if and only if they are twin vertices in \( \overline{G} \). If \( S \) is a resolving set of either \( G \) or \( \overline{G} \), then \( S \cap \{ u, w \} \neq \emptyset \).

**Observation 2.2.** Graphs \( G \) and \( \overline{G} \) cannot both be disconnected.

**Observation 2.3.** [15] For any family \( \mathcal{C} = \{ H_1, \ldots, H_t \} \),

\[
\max_{1 \leq i \leq t} \{ \dim(H_i) \} \leq Sd(\mathcal{C}) \leq \min \left\{ \sum_{i=1}^t \dim(H_i), |V(G)| - 1 \right\}.
\]

Observation 2.3 implies the following result.

**Corollary 2.4.** For any graph \( G \) of order \( n \geq 2 \),

\[
1 \leq \max \{ \dim(G), \dim(\overline{G}) \} \leq Sd(G, \overline{G}) \leq \min \{ \dim(G) + \dim(\overline{G}), n - 1 \} \leq n - 1.
\]

Next, we characterize graphs \( G \) satisfying \( Sd(G, \overline{G}) \) equals 1 and \( |V(G)| - 1 \), respectively. We begin by recalling some known results. In the following, \( G + H \) denotes the join of graphs \( G \) and \( H \), which is obtained from the disjoint union of \( G \) and \( H \) by adding an edge \( xy \) for each \( x \in V(G) \) and each \( y \in V(H) \).

**Theorem 2.5.** [4] Let \( G \) be a connected graph \( G \) of order \( n \geq 2 \). Then

(a) \( \dim(G) = 1 \) if and only if \( G = P_2 \);

(b) for \( n \geq 4 \), \( \dim(G) = n - 2 \) if and only if \( G = K_{s,t} \) (\( s, t \geq 1 \)), \( G = K_s + \overline{K}_t \) (\( s \geq 1, t \geq 2 \)), or \( G = K_s + (K_1 \cup K_1) \) (\( s, t \geq 1 \));

(c) \( \dim(G) = n - 1 \) if and only if \( G = K_n \).

**Theorem 2.6.** [15] Let \( \mathcal{C} \) be a family of connected graphs on a common vertex set. Then

(a) \( Sd(\mathcal{C}) = 1 \) if and only if \( \mathcal{C} \) is a collection of paths that share a common end vertex.

(b) If \( \mathcal{C} \) is a collection of paths, then \( Sd(\mathcal{C}) \in \{1, 2\} \).

**Theorem 2.7.** Let \( G \) be a graph of order \( n \geq 2 \). Then

(a) \( Sd(G, \overline{G}) = 1 \) if and only if \( G \in \{ P_2, P_3, P_3, P_3 \} \);

(b) \( Sd(G, \overline{G}) = n - 1 \) if and only if \( G \in \{ K_n, \overline{K}_n \} \);

Proof. Let \( G \) be a graph of order \( n \geq 2 \).

(a) \( \iff \) If \( G = P_2 \) is given by \( u_1u_2 \), then \( \{ u_1 \} \) forms a resolving set for both \( G \) and \( \overline{G} \); notice that \( d_G(u_1, u_2) = 1 \) and \( d_{\overline{G}}(u_1, u_2) = \infty \). If \( G = P_3 \) is given by \( u_1u_2u_3 \), then \( \overline{G} = K_1 \cup K_2 \) consists of two components, \( u_2 \) and \( u_1u_3 \), and \( \{ u_1 \} \) forms a resolving set for both \( G \) and \( \overline{G} \); note that \( d_G(u_1, u_2) = 1 < 2 = d_G(u_1, u_3) \) and \( d_{\overline{G}}(u_1, u_1) = 1 < \infty = d_{\overline{G}}(u_1, u_2) \).

\( \implies \) Let \( Sd(G, \overline{G}) = 1 \). We may, by Observation 2.2, assume \( G \) to be connected; then \( G = P_n \) by Theorem 2.5(a). If \( n \geq 4 \), then \( \overline{G} \) is connected but not a path which shares a common end vertex with \( G \). By Theorem 2.6(a), \( Sd(G, \overline{G}) \neq 1 \).

(b) \( \iff \) Let \( G = K_n \). Since any two vertices of \( G \) (and thus \( \overline{G} \)) are twin vertices, \( Sd(G, \overline{G}) \geq n - 1 \) by Observation 2.1. On the other hand, \( Sd(G, \overline{G}) \leq n - 1 \) by Corollary 2.4. Thus, \( Sd(G, \overline{G}) = n - 1 \).

\( \implies \) Let \( Sd(G, \overline{G}) = n - 1 \). The case \( n = 2 \) is subsumed by Part (a) of this proof. So, let \( n \geq 3 \) and assume, to the contrary, that \( G \not\in \{ K_n, \overline{K}_n \} \). Then, there exist three distinct vertices, say \( x, y, z \in V(G) \), such that \( xy \in E(G) \) and \( zx \notin E(G) \). Then \( V(G) - \{ y, z \} \) is a resolving set for both \( G \) and \( \overline{G} \), since \( d_G(x, y) = 1 < d_G(x, z) \) and \( d_{\overline{G}}(x, z) = 1 < d_{\overline{G}}(x, y) \). So, \( G \not\in \{ K_n, \overline{K}_n \} \) implies \( Sd(G, \overline{G}) \leq n - 2 \) for \( n \geq 3 \). \(\square\)
Next, we show that if \( \{ \text{diam}(G), \text{diam}(\overline{G}) \} \neq \{3\} \), then \( \text{Sd}(G, \overline{G}) = \max\{\text{dim}(G), \text{dim}(\overline{G})\} \). We begin with the following lemma.

**Lemma 2.8.** Let \( G \) be a graph with \( \text{diam}(G) = 2 \). If \( S \) is a resolving set for \( G \), then \( S \) is a resolving set for \( \overline{G} \).

**Proof.** Let \( S \subseteq V(G) \) be a resolving set of \( G \); note that vertices in \( S \) are self-resolved in \( G \) and \( \overline{G} \). Suppose there are distinct vertices \( x \) and \( y \) in \( V(G) \) such that \( \text{d}_G(x, y) = \text{d}_G(y, x) = 1 \). Then, there exists \( z \in S \) such that \( \text{d}_G(x, z) = 1 \) and \( \text{d}_G(y, z) = 2 \). So, \( z \in S \) resolves \( x \) and \( y \) in \( G \), then \( z \) resolves \( x \) and \( y \) in \( \overline{G} \). \( \square \)

**Proposition 2.9.** [2] If \( G \) is a graph with \( \text{diam}(G) \geq 4 \), then \( \text{diam}(\overline{G}) \leq 2 \).

**Theorem 2.10.** Let \( G \) be a graph of order at least two. Suppose \( \{ \text{diam}(G), \text{diam}(\overline{G}) \} \neq \{3\} \) and \( \text{diam}(G) \leq \text{diam}(\overline{G}) \); then \( \text{Sd}(G, \overline{G}) = \text{dim}(G) \).

**Proof.** Let \( G \) be a graph of order \( n \geq 2 \). By hypotheses and Proposition 2.9, we find \( \text{diam}(G) \leq 2 \). If \( \text{diam}(G) = 1 \), then \( G = K_n \) and \( \text{Sd}(G, \overline{G}) = n - 1 = \text{dim}(G) = \text{dim}(\overline{G}) \) by Theorem 2.7(b). If \( \text{diam}(G) = 2 \), then \( \text{Sd}(G, \overline{G}) = \text{dim}(G) = \max\{\text{dim}(G), \text{dim}(\overline{G})\} \) by Lemma 2.8. \( \square \)

We note that Theorem 2.10 implies that \( \text{Sd}(G, \overline{G}) = \text{dim}(G) \) when \( G \) is the Petersen graph, a complete multi-partite graph, a wheel graph \( K_1 + C_{n-1}, n \geq 4 \), or a fan graph \( K_1 + P_{n-1}, n \geq 3 \), since those graphs have diameter 2.

### 3 Some graphs \( G \) with \( \text{diam}(G) = 3 = \text{diam}(\overline{G}) \)

In this section, we consider graphs \( G \) with \( \text{diam}(G) = 3 = \text{diam}(\overline{G}) \). First, we determine \( \text{Sd}(T, \overline{T}) \) for tress \( T \) with \( \text{diam}(T) = 3 = \text{diam}(\overline{T}) \).

We recall some terminology. Fix a tree \( T \). An end vertex \( \ell \) is called a *terminal vertex* of a major vertex \( v \) if \( \text{d}_T(\ell, v) < \text{d}_T(\ell, w) \) for every other major vertex \( w \) in \( T \). The *terminal degree*, \( \text{t}_T(v) \), of a major vertex \( v \) is the number of terminal vertices of \( v \) in \( T \), and an *exterior major vertex* is a major vertex that has a positive terminal degree. We denote by \( \text{ex}(T) \) the number of exterior major vertices of \( T \), and \( \sigma(T) \) the number of end vertices of \( T \). Let \( M(T) \) be the set of exterior major vertices of \( T \). Let \( M_1(T) = \{ w \in M(T) : \text{t}_T(w) = 1 \} \) and let \( M_2(T) = \{ w \in M(T) : \text{t}_T(w) \geq 2 \} \); note that \( M(T) = M_1(T) \cup M_2(T) \). For each \( v \in M(T) \), let \( T_v \) be the subtree of \( T \) induced by \( v \) and all vertices belonging to the paths joining \( v \) with its terminal vertices, and let \( L_v(T) \) be the set of terminal vertices of \( v \) in \( T \).

**Theorem 3.1.** [4, 10, 11] For any tree \( T \) that is not a path, \( \text{dim}(T) = \sigma(T) - \text{ex}(T) \).

**Theorem 3.2.** [11] Let \( T \) be a tree with \( \text{ex}(T) = k \geq 1 \), and let \( v_1, v_2, \ldots, v_k \) be the exterior major vertices of \( T \). For each \( i \in [k] \), let \( \ell_{i,1}, \ell_{i,2}, \ldots, \ell_{i,(\sigma_i - 1)} \) be the terminal vertices of \( v_i \) with \( \text{t}_T(v_i) = \sigma_i \geq 1 \), and let \( P_{i,j} \) be the \( v_i - \ell_{i,j} \) path, where \( j \in [\sigma_i] \). Let \( W \subseteq V(T) \). Then \( W \) is a basis of \( T \) if and only if \( W \) contains exactly one vertex from each of the paths \( P_{i,j} - v_i \), where \( j \in [\sigma_i] \) and \( i \in [k] \), with exactly one exception for each \( i \in [k] \) and \( W \) contains no other vertices of \( T \).

**Proposition 3.3.** Let \( T \) be a tree of order \( n \) with \( \text{diam}(T) = 3 = \text{diam}(\overline{T}) \). Then \( T \) satisfies one of the following:

(a) \( T = P_4 \);

(b) \( \text{ex}(T) = 1 \) and \( \sigma(T) = n - 2 \);

(c) \( \text{ex}(T) = 2 \) and \( \sigma(T) = n - 2 \).
Moreover, \( S_d(T, \overline{T}) = \left\{ \begin{array}{ll} 2 = \dim(T) + \dim(\overline{T}) & \text{if } T \text{ satisfies (a)}, \\ n - 3 = \dim(T) = \dim(\overline{T}) & \text{if } T \text{ satisfies (b)}, \\ n - 3 = \dim(T) = \dim(\overline{T}) + 1 & \text{if } T \text{ satisfies (c)}. \end{array} \right. \)

Proof. Let \( T \) be a tree of order \( n \geq 4 \) with \( \dim(T) = 3 \). Then \( \text{ex}(T) \in \{0, 1, 2\} \), since \( \text{ex}(T) \geq 3 \) implies \( \dim(T) \geq 4 \).

First, let \( \text{ex}(T) = 0 \). Then \( \dim(T) = 3 \) implies \( T = P_4 \) (see Figure 1(a)), and \( \overline{T} = P_4 \). So, \( S_d(P_4, \overline{P_4}) = 2 \) by Theorem 2.6.

Second, let \( \text{ex}(T) = 1 \). Let \( v \) be the exterior major vertex of \( T \), and let \( \ell_1, \ell_2, \ldots, \ell_k \) be the terminal vertices of \( v \) in \( T \) such that \( d_G(v, \ell_1) \geq d_G(v, \ell_2) \geq \cdots \geq d_G(v, \ell_k) \), where \( k \geq 3 \). Note that \( \dim(T) = 3 \) implies \( d_G(v, \ell_1) = 2 = 1 + d_G(v, \ell_2) \); see Figure 1(b). If \( s \) is a degree two vertex lying on the \( v-\ell_1 \) path in \( T \), then \( d_T(v, s) = 3 \) and it is easy to see that \( \dim(T) = 3 \). Since \( S_1 = \bigcup_{i=1}^{k-1} \{ \ell_i \} \) forms a resolving set for both \( T \) and \( \overline{T} \) with \( |S_1| = k-1 = n-3 \), \( S_d(T, \overline{T}) \leq n-3 \). By Corollary 2.4 and Theorem 3.1, \( S_d(T, \overline{T}) \geq \dim(T) = k-1 = n-3 \); thus \( S_d(T, \overline{T}) = n-3 = \dim(T) = \dim(\overline{T}) \).

Third, let \( \text{ex}(T) = 2 \). Let \( v_1 \) and \( v_2 \) be distinct exterior major vertices of \( T \), let \( \ell_1, \ldots, \ell_a \) be the terminal vertices of \( v_1 \) and let \( \ell'_1, \ldots, \ell'_b \) be the terminal vertices of \( v_2 \) in \( T \), where \( a, b \geq 2 \). Note that the condition \( \dim(T) = 3 \) implies that \( d_T(v_1, v_2) = 1 \) and \( d_T(v_1, \ell_i) = 1 = d_T(v_2, \ell'_j) \), where \( i \in [a] \) and \( j \in [b] \) (see Figure 1(c)). Let \( S \) be any resolving set for \( T \). Then \( |S \cap (\bigcup_{i=1}^{a} \{ \ell_i \})| \geq a-1 \) and \( |S \cap (\bigcup_{j=1}^{b} \{ \ell'_j \})| \geq b-1 \) by Observation 2.1. If \( S_2 = \bigcup_{i=1}^{a} \{ \ell_i \} \cup \bigcup_{j=1}^{b} \{ \ell'_j \} \), then \( S_2 \) forms a resolving set for \( T \) with \( |S_2| = \dim(T) \) by Theorems 3.1 and 3.2, and \( \text{code}_{S_2}(\ell_a) = \text{code}_{S_2}(\ell'_b) \).

So, \( |S| \geq |S_2| + 1 = a + b - 1 = n - 3 = \dim(T) + 1 \). Since \( S_2 \cup \{ \ell_a \} \) forms a resolving set for \( T \) and \( \overline{T} \), \( \dim(T) \leq n-3 \); thus \( \dim(T) = n-3 \) and \( S_d(T, \overline{T}) = n-3 = \dim(T) = \dim(\overline{T}) + 1 \).

![Figure 1: Trees T with \( \dim(T) = 3 = \dim(\overline{T}) \).](image)

Second, we determine \( S_d(G, \overline{G}) \) for unicyclic graphs \( G \) with \( \dim(G) = 3 = \dim(\overline{G}) \). We begin with the characterization of unicyclic graphs \( G \) satisfying \( \dim(G) = 3 \).

**Lemma 3.4.** Let \( G \) be a unicyclic graph with \( \dim(G) = 3 \). Then \( G \in \{C_6, C_7\} \) or \( G \) is isomorphic to a graph represented in Figure 2.

![Figure 2: Unicyclic graphs \( G \not\in \{C_6, C_7\} \) with \( \dim(G) = 3 \), where \( a, b, c \geq 1 \).](image)

Proof. Let \( G \) be a unicyclic graph with \( \dim(G) = 3 \). Let \( C = C_m \) be the unique cycle of \( G \) given by \( u_1, u_2, \ldots, u_m, u_1 \), where \( m \geq 3 \). For \( w \in V(C) \), let \( T_w \) denote the subtree rooted at \( w \) in \( G \) and let \( \tau(G) = \{ u_i \in V(C) : \deg_G(u_i) \geq 3 \} \) and let \( L_i(G) = \{ \ell \in V(T_{u_i}) : \deg_G(\ell) = 1 \} \).

**Case 1:** \( \tau(G) = \emptyset \). In this case, \( G \) is a cycle and \( \dim(G) = 3 \) implies \( G \in \{C_6, C_7\} \).
Case 2: $\tau(G) \neq \emptyset$. In this case, $m \in \{3, 4, 5\}$; note that $m \geq 6$ implies $\text{diam}(G) \geq 4$. By relabeling the vertices of $G$ in $C$ if necessary, let $\text{deg}_G(u_1) \geq 3$.

Subcase 2.1: $m = 3$. First, suppose $\text{deg}_G(u_2) = \text{deg}(u_3) = 2$. In this case, $d_G(\ell, u_1) \leq 2$ for each $\ell \in L_1(G)$, and there exists an end-vertex $\ell' \in L_1(G)$ with $d_G(\ell', u_1) = 2$. So, $G$ is isomorphic to Figure 2(a), where $a \geq 1$ and $b \geq 1$.

Second, suppose $\text{deg}_G(u_2) \geq 3$ or $\text{deg}_G(u_3) \geq 3$. If $\text{deg}_G(u_i) \geq 3$ and $\ell_i \in L_1(G)$, then $d_G(\ell_i, u_1) = 1$, where $i \in [3]$. If $\text{deg}_G(u_2) \geq 3 = 1 + \text{deg}_G(u_3)$ or $\text{deg}_G(u_3) \geq 3 = 1 + \text{deg}_G(u_2)$, then $G$ is isomorphic to Figure 2(b), where $a, b \geq 1$. If $\text{deg}_G(u_2) \geq 3$ and $\text{deg}_G(u_3) \geq 3$, then $G$ is isomorphic to Figure 2(c), where $a, b, c \geq 1$.

Subcase 2.2: $m \in \{4, 5\}$. We note the following: (i) for each end-vertex $\ell_i \in L_1(G)$, $d_G(\ell_i, u_1) = 1$, where $i \in [m]$; (ii) $\text{deg}_G(u_3) = 2 = \text{deg}_G(u_{m-1})$. If every vertex in $\{u_1, u_2, u_m\}$ has degree at least three in $G$, say $\ell_j$ is a terminal vertex of the major vertex $u_i$ for each $j \in \{1, 2, m\}$, then $d_G(\ell_2, \ell_m) \geq 4$, and thus $\text{diam}(G) \geq 4$. So, $\text{deg}_G(u_1) \geq 3$ and $\text{diam}(G) = 3$ implies that at most one vertex in $\{u_2, u_m\}$ has degree at least three in $G$. If $\text{deg}_G(u_2) = 2 = \text{deg}_G(u_m)$, then $G$ is isomorphic to Figure 2(d) (when $m = 4$) or $G$ is isomorphic to Figure 2(f) (when $m = 5$), where $a \geq 1$. If $\text{deg}_G(u_2) \geq 3$ or $\text{deg}_G(u_m) \geq 3$, but not both, then $G$ is isomorphic to Figure 2(e) (when $m = 4$) or $G$ is isomorphic to Figure 2(g) (when $m = 5$), where $a, b \geq 1$.

**Proposition 3.5.** Let $G$ be a unicyclic graph of order $n$ with $\text{diam}(G) = 3 = \text{diam}(\overline{G})$. Then $G$ is isomorphic to Figure 2(a)-(b) or Figure 2(d)-(e). Moreover, $\text{Sd}(G, \overline{G})$ equals

$$\begin{align*}
\text{dim}(G) + 1 &= \text{dim}(G) + 1 \quad \text{if } G \text{ is isomorphic to Fig. 2(a) with } a = b = 1; \\
\text{dim}(G) &= \text{dim}(G) \quad \text{if } G \text{ is isomorphic to Fig. 2(a) with } a = 1 \text{ and } b \geq 1, \\
\text{dim}(G) &= \text{dim}(G) + 1 \quad \text{if } G \text{ is isomorphic to Fig. 2(b) with } a = 1, \\
\text{dim}(G) &= \text{dim}(G) + 1 \quad \text{if } G \text{ is isomorphic to Fig. 2(d) with } a \geq 1, \\
\text{dim}(G) &= \text{dim}(G) + 1 \quad \text{if } G \text{ is isomorphic to Fig. 2(e) with } a \geq 2 = 1 + b \text{ or } b \geq 2 = 1 + a, \\
\text{dim}(G) &= \text{dim}(G) + 2 \quad \text{if } G \text{ is isomorphic to Fig. 2(b) with } a, b \geq 2.
\end{align*}$$

Proof. Let $G$ be a unicyclic graph of order $n$ with $\text{diam}(G) = 3$. By Lemma 3.4, $G \in \{C_6, C_7\}$ or $G$ is isomorphic to one of the graphs in Figure 2. If $G \in \{C_6, C_7\}$, then $\text{dim}(\overline{G}) = 2$. If $G$ is isomorphic to Figure 2(a), then $\text{dim}(\overline{G}) = 3$; note that $d_{\overline{G}}(u_1, s_1) = 3$. If $G$ is isomorphic to Figure 2(b) or Figure 2(d)-(e), then $\text{dim}(\overline{G}) = 3$; note that $d_{\overline{G}}(u_1, u_2) = 3$. If $G$ is isomorphic to Figure 2(c) or Figure 2(f)-(g), one can easily check that $\text{diam}(\overline{G}) = 2$.

Next, for unicyclic graphs $G$ with $\text{diam}(G) = 3 = \text{diam}(\overline{G})$, we determine $\text{Sd}(G, \overline{G})$. Let $S, \overline{S}$, respectively, be any basis for $G, \overline{G}$, respectively, such that $S_0 = S \cap \overline{S}$ has the maximum cardinality.

Case 1: $G$ is isomorphic to Figure 2(a). By Observation 2.1, we have the following: (i) if $a \geq 2$, then $|S_0 \cap \cup_{i=1}^a \{\ell_i\}| \geq a - 1$; (ii) $|S_0 \cap \{u_2, u_3\}| \geq 1$; (iii) if $b \geq 3$, then $|S_0 \cap \cup_{i=2}^b \{s_i\}| \geq b - 2$.

First, suppose $a = 1$ and $b \in \{1, 2\}$. Then $\{u_2, s_1\}$ forms a basis for $G$ and $\overline{G}$, and thus $\text{Sd}(G, \overline{G}) = \text{dim}(\overline{G}) = 2 = \text{dim}(G)$; note that $\text{Sd}(G, \overline{G}) = n - 3$ if $a = b = 1$, and $\text{Sd}(G, \overline{G}) = n - 4$ if $a = 1$ and $b = 2$.

Second, suppose $a = 1$ and $b \geq 3$. From (ii) and (iii), we may assume that $R_0 = \{u_2\} \cup \cup_{i=3}^b \{s_i\} \subseteq S_0$ with $|R_0| = b - 1 = n - 5$. Since $\text{code}_{G, R_0}(s_1) = \text{code}_{G, R_0}(s_2)$ and $\text{code}_{\overline{G}, R_0}(s_1) = \text{code}_{\overline{G}, R_0}(s_2)$, $|S| \geq |R_0| + 1$ and $|\overline{S}| \geq |R_0| + 1$. On the other hand, $R_0 \cup \{s_1\}$ forms a resolving set for $G$ and $\overline{G}$, and hence $|S| \leq |R_0| + 1$ and $|\overline{S}| \leq |R_0| + 1$. Thus $\text{Sd}(G, \overline{G}) = \text{dim}(\overline{G}) = n - 4 = \text{dim}(G)$.

Third, suppose $a \geq 2$ and $b \in \{1, 2\}$. From (i) and (ii), we may assume that $R_1 = \{u_2\} \cup \cup_{i=2}^b \{\ell_i\} \subseteq S_0$ with $|R_1| = a$. Then $R_1$ forms a resolving set for $G$, but $\text{code}_{\overline{G}, R_1}(u_1) = \text{code}_{\overline{G}, R_1}(u_3)$ and $R_1 \cup \{s_1\}$ forms a resolving set for $\overline{G}$. Thus, $\text{Sd}(G, \overline{G}) = \text{dim}(\overline{G}) = a + 1 = \text{dim}(G) + 1$; we note that $\text{Sd}(G, \overline{G}) = \text{dim}(\overline{G}) = n - 3 = \text{dim}(G) + 1$ if $a \geq 2$ and $b = 1$, and $\text{Sd}(G, \overline{G}) = \text{dim}(\overline{G}) = n - 4 = \text{dim}(G) + 1$ if $a \geq 2$ and $b = 2$. 

6
Fourth, suppose $a \geq 2$ and $b \geq 3$. From (i), (ii) and (iii), we may assume that $R_2 = \{u_2\} \cup (\cup_{i=2}^{a}(\ell_i)) \cup (\cup_{i=3}^{b}(s_i)) \subseteq S_0$ with $|R_2| = a + b - 2 = n - 5$. Then $R_2$ forms a resolving set for $G$, but $code_{\overline{G},R_2}(\ell_1) = code_{\overline{G},R_2}(s_2)$ and $R_2 \cup \{s_1\}$ forms a resolving set for $\overline{G}$; thus $Sd(G,\overline{G}) = dim(\overline{G}) = n - 4 = dim(G) + 1$.

Case 2: $G$ is isomorphic to Figure 2(b). First, suppose $a = b = 1$. Then $\{\ell_1, \ell_1'\}$ forms a basis for $G$ and $\overline{G}$; thus $Sd(G,\overline{G}) = dim(\overline{G}) = dim(G) = 2 = n - 3$.

Second, suppose $a \geq 2$ and $b = 1$, or $a = 1$ and $b \geq 2$, say the former by relabeling the vertices of $G$ if necessary; then $|S_0 \cap (\cup_{i=1}^{a}(\ell_i))| \geq a - 1$ by Observation 2.1. We may assume that $W_0 = \cup_{i=2}^{a}(\ell_i) \subseteq S_0$ with $|W_0| = a - 1 = n - 5$. Then $code_{G,W_0}(\ell_1) = code_{G,W_0}(u_2) = code_{\overline{G},W_0}(\ell_1) = code_{\overline{G},W_0}(u_2) = code_{\overline{G},W_0}(u_3)$; moreover, for any $v \in V(G) - W_0$, $W_0 \cup \{v\}$ fails to be a resolving set for $\overline{G}$. So, $dim(G) \geq |W_0| + 1$ and $dim(\overline{G}) \geq |W_0| + 2$. Since $W_0 \cup \{u_2\}$ forms a resolving set for $G$ and $W_0 \cup \{u_2, u_3\}$ forms a resolving set for $\overline{G}$ (as well as for $G$), $dim(G) \leq |W_0| + 1 = n - 4$ and $dim(\overline{G}) \leq |W_0| + 2 = n - 3$. Thus, $Sd(G,\overline{G}) = dim(\overline{G}) = n - 3 = dim(G) + 1$.

Third, suppose $a \geq 2$ and $b \geq 2$; then $|S_0 \cap (\cup_{i=1}^{a}(\ell_i))| \geq a - 1$ and $|S_0 \cap (\cup_{i=2}^{b}(s_i))| \geq b - 1$ by Observation 2.1. We may assume that $W_1 = (\cup_{i=2}^{a}(\ell_i)) \cup (\cup_{i=2}^{b}(s_i)) \subseteq S_0$ with $|W_1| = a + b - 2 = n - 5$. Then $W_1$ is a resolving set for $G$, but $code_{\overline{G},W_1}(\ell_1) = code_{\overline{G},W_1}(s_1)$ and $W_1 \cup \{v\}$ fails to be a resolving set for $\overline{G}$ for any $v \in V(G) - W_1$. So, $dim(G) = |W_1| = n - 5$ and $dim(\overline{G}) \geq |W_1| + 2$. Since $W_1 \cup \{u_2, u_3\}$ forms a resolving set for $\overline{G}$, $dim(G) \leq |W_1| + 2$. Thus, $Sd(G,\overline{G}) = dim(\overline{G}) = n - 3 = dim(G) + 2$.

Case 3: $G$ is isomorphic to Figure 2(d). If $a = 1$, then $\{u_2, u_3\}$ forms a basis for $G$ and $\overline{G}$; thus, $Sd(G,\overline{G}) = dim(\overline{G}) = dim(G) = 2 = n - 3$. So, suppose $a \geq 2$; then $|S_0 \cap (\cup_{i=1}^{a}(\ell_i))| \geq a - 1$ and $|S_0 \cap \{u_2, u_4\}| \geq 1$ by Observation 2.1. We may assume that $R = \{u_2\} \cup (\cup_{i=2}^{a}(\ell_i)) \subseteq S_0$ with $|R| = a = n - 4$. Then $code_{G,R}(\ell_1) = code_{\overline{G},R}(u_4)$ and $code_{\overline{G},R}(\ell_1) = code_{\overline{G},R}(u_3)$; thus, $dim(G) \geq |R| + 1$ and $dim(\overline{G}) \geq |R| + 1$. Since $R \cup \{u_4\}$ forms a resolving set for both $G$ and $\overline{G}$, $dim(G) \leq |R| + 1$ and $dim(\overline{G}) \leq |R| + 1$. So, $Sd(G,\overline{G}) = dim(\overline{G}) = n - 3$.

Case 4: $G$ is isomorphic to Figure 2(e). First, suppose $a = b = 1$. It is easy to check that $dim(G) = 2 = dim(G)$, but neither $S$ nor $\overline{S}$ forms a resolving set for $Sd(G,\overline{G})$. There are exactly four following bases for $G$: $\{\ell_1, \ell_1', \ell_1, u_3\}, \{\ell_1', u_4\}, \{u_2, u_3\}$ and $\{u_3, u_4\}$. Similarly, there are exactly four following bases for $\overline{G}$: $\{u_1, u_4\}, \{u_1, u_4, u_3\}, \{u_2, u_3\}$ and $\{u_1, u_3, u_4\}$. Since $\{u_1, u_3, u_4\}$ forms a resolving set for both $G$ and $\overline{G}$, $Sd(G,\overline{G}) = dim(\overline{G}) = 3 = n - 3$.

Second, suppose $a \geq 2$ and $b = 1$, or $a = 1$ and $b \geq 2$, say the former by relabeling the vertices of $G$ if necessary; then $|S_0 \cap (\cup_{i=1}^{a}(\ell_i))| \geq a - 1$ by Observation 2.1. We may assume that $W_0 = \cup_{i=2}^{a}(\ell_i) \subseteq S_0$ with $|W_0| = a - 1 = n - 6$. Note that, for any $v \in V(G) - W_0$, $W_0 \cup \{v\}$ fails to be a resolving set for either $G$ or $\overline{G}$; thus, $|S| \geq |W_0| + 2$ and $|\overline{S}| \geq |W_0| + 2$. Since $W_0 \cup \{u_2, u_3\}$ forms a resolving set for both $G$ and $\overline{G}$, $|S| \leq |W_0| + 2$ and $|\overline{S}| \leq |W_0| + 2$. Thus, $Sd(G,\overline{G}) = dim(\overline{G}) = n - 4 = dim(G)$.

Third, suppose $a \geq 2$ and $b \geq 2$; then $|S_0 \cap (\cup_{i=1}^{a}(\ell_i))| \geq a - 1$ and $|S_0 \cap (\cup_{i=2}^{b}(s_i))| \geq b - 1$ by Observation 2.1. We may assume that $W = (\cup_{i=2}^{a}(\ell_i)) \cup (\cup_{i=2}^{b}(s_i)) \subseteq S_0$ with $|W| = a + b - 2 = n - 6$. Then $code_{G,W}(\ell_1) = code_{G,W}(u_4)$, $code_{G,W}(\ell_1') = code_{G,W}(u_3)$, and $code_{\overline{G},W}(\ell_1) = code_{\overline{G},W}(\ell_1') = code_{\overline{G},W}(u_4)$; moreover, for any $v \in V(G) - W$, $W \cup \{v\}$ fails to be a resolving set for $\overline{G}$. So, $dim(G) \geq |W| + 1$ and $dim(\overline{G}) \geq |W| + 2$. Since $W \cup \{u_3\}$ forms a resolving set for $G$ and $W \cup \{u_3, \ell_1'\}$ forms a resolving set for $\overline{G}$, $dim(G) \leq |W| + 1$ and $dim(\overline{G}) \leq |W| + 2$. Thus, $Sd(G,\overline{G}) = dim(\overline{G}) = n - 4 = dim(G)$.

A graph $G$ is a split graph if $V(G)$ can be partitioned into a clique and an independent set. It is easy to see that the complement of a split graph is a split graph. A self-complementary graph is a graph that is isomorphic to its own complement. Next, we construct a family of self-complementary split graphs $G$ with $diam(G) = 3$. For each integer $k \geq 1$, let $G_k$ be a graph of order $4k$ with $V(G_k) = V_1 \cup V_2$, where $|V_1| = 2k = |V_2|$, such that the edge set of $G_k$ is specified as follows: (i) $V_1 = \cup_{i=0}^{2k-1}\{a_i\}$ induces $K_{2k}$ in $G_k$; (ii) $V_2 = \cup_{i=0}^{2k-1}\{b_i\}$ induces $K_{2k}$ in $G_k$; (iii) for each $i \in [2k-1]$,
$a_i$ is adjacent to $b_i, \ldots, b_{i+k-1}$, where the subscript is taken modulo $2k$; (iv) there are no other edges. See Figure 3 for $G_3$ and the labeling of its vertices.

We recall the concept of a resolving function, which will be used in proving Theorem 3.6. Let $g$ be a function on $V(G)$ with codomain $[0, 1]$; for $S \subseteq V(G)$, let $g(S) = \sum_{u \in S} g(u)$. If $g(RG\{x, y\}) \geq 1$ for any distinct $x, y \in V(G)$, then $g$ is called a resolving function of $G$. We note that $\dim(G) \geq \min\{g(V(G)) : g$ is a resolving function of $G\}$; in fact, the preceding inequality would be an equality if the codomain of $g$ is taken to be $\{0, 1\}$ in lieu of $[0, 1]$.

![Figure 3: $G_3$ satisfying $\text{Sd}(G_3, \overline{G_3}) > 3 = \dim(G_3) = \dim(\overline{G_3})$.](image)

**Theorem 3.6.** For $k \geq 1$, let $G_k$ be the family of split graphs defined above. Then we have the following:

(a) $G_k$ is a self-complementary graph with $\dim(G_k) = 3 = \dim(\overline{G_k})$;

(b) $\dim(G_k) = \dim(\overline{G_k}) = k$;

(c) $\text{Sd}(G_k, \overline{G_k}) \leq 2k - 1$ for $k \geq 2$;

(d) $\text{Sd}(G_k, \overline{G_k}) > k$.

**Proof.** Let $k \geq 1$.

(a) First, we show that $G_k$ is self-complementary. Let $\phi : V(G_k) \to V(\overline{G_k})$ be a map such that $\phi(a_i) = b_i$ and $\phi(b_i) = a_{i+1}$, where the subscript is taken modulo $2k$. It is easily checked that $\phi$ is a graph isomorphism. Second, we show that $\dim(G_k) = 3 = \dim(\overline{G_k})$. Since $G_k$ is self-complementary, it suffices to show that $\dim(G_k) = 3$. We note the following: (i) $d_G(a_i, a_j) = 1$ for any distinct $i, j \in [2k - 1]$; (ii) $d_G(a_i, b_j) \in \{1, 2\}$ for any $i, j \in [2k - 1]$; (iii) $d_G(b_i, b_j) \in \{2, 3\}$ for any distinct $i, j \in [2k - 1]$.

(b) Since $G_k$ is isomorphic to $\overline{G_k}$, it suffices to show that $\dim(G_k) = k$. First, we show that $\dim(G_k) \leq k$. Let $S = \{b_0, b_1, \ldots, b_{k-1}\}$ with $|S| = k$. We note the following: (i) $\text{code}_{G_k,S}(a_0)$ has 1 in all of its entries; (ii) for $i \in [k - 1]$, $\text{code}_{G_k,S}(a_i)$ has 2 in the first $i$ entries and 1 in the rest of its entries; (iii) $\text{code}_{G_k,S}(a_k)$ has 2 in all of its entries; (iv) for $j \in [k - 1]$, $\text{code}_{G_k,S}(a_{k+j})$ has 1 in the first $j$ entries and 2 in the rest of its entries; (v) for $i \in [k - 1]$, $\text{code}_{G_k,S}(b_{i+k})$ has 3 in the $(i + 1)$th entry and 2 in the rest of its entries; (vi) for $i \in [k - 1]$, $\text{code}_{G_k,S}(b_i)$ has 0 in the $i$th entry and 2 in the rest of its entries. Thus, $S$ is a resolving set for $G_k$, and thus $\dim(G_k) \leq k$. Next, we show that $\dim(G_k) \geq k$. Let $g : V(G_k) \to [0, 1]$ be a minimum resolving function of $G_k$. Since $RG_k\{a_i, a_{i+1}\} = \{a_i, a_{i+1}, b_i, b_{i+k}\}$, we have $g(RG_k\{a_i, a_{i+1}\}) = g(a_i) + g(a_{i+1}) + g(b_i) + g(b_{i+k}) \geq 1$ for each $i \in [2k - 1]$, where the subscript is taken modulo $2k$. By summing over $2k$ such inequalities, we have $2 \sum_{i=0}^{2k-1} (g(a_i) + g(b_i)) \geq 2k$, i.e., $g(V(G_k)) \geq k$. So, $\dim(G_k) \geq k$. 

8
(c) Let $k \geq 2$ for this part. It is enough to show that $S = V_1 - \{a_{2k-1}\}$ resolves both $G_k$ and $\overline{G}_k$. Clearly, $S$ resolves $\overline{G}_k$ by part (b). With respect to resolving $G_k$, notice (1) each pair of vertices of $V_1$ is resolved by $S$; (2) each vertex $x \in V_1$ is resolved from any vertex $y \in V_2$ by $S$ since code$_{G_k,S}(x)$ does not contain 2 as a component, whereas code$_{G_k,S}(y)$ does; (3) the map code$_{G_k,S'}$ is already injective on $V_2$, where $S' = \{a_0, \ldots, a_{k-1}\} \subseteq S$.

(d) Since $G_1 = P_4$, Sd($G_1, \overline{G}_1) = 2 > 1$ by Theorem 2.6. Put $V = V(G_k) = V(\overline{G}_k)$. Let $k \geq 2$ and $S \subseteq V$ be a basis of $G_k$ with $|S_2| = |S \cap V_2| = \beta \leq k$; we will show that $\beta = k$. Consider the partition $P$ of $V_1$ given by the map code$_{G_k,S}$; i.e., two vertices of $V_1$ belong to the same cell of $P$ exactly when they are mapped to the same vector under code$_{G_k,S}$. By the adjacency relation of $b_i \in V_2(G_k)$, it is easily seen that $|P| = 2\beta$. Since the subgraph induced by $V_1$ in $G_k$ is a clique, all but one vertex in each cell $C_i$ of $P$ must belong to $S$. Thus, we have the inequality $\sum_{i=1}^{2\beta} (|C_i| - 1) \leq k - \beta$. Since the left side of the preceding inequality equals $2(k - \beta)$, we find $\beta = k$. It follows that a basis of $G_k$ intersects trivially with $V_1$ and thus does not resolve $\overline{G}_k$. By symmetry, a basis of $\overline{G}_k$ contains no vertex of $V_2$ and thus cannot resolve $G_k$. Therefore, Sd($G_k, \overline{G}_k) \geq k + 1$ for $k \geq 1$.

Noting Sd($G_1, \overline{G}_1) = 2$ and Sd($G_2, \overline{G}_2) = 3$ (as is easily checked), there is some empirical evidence to suggest the following.

Conjecture 3.7. For $k \geq 3$, Sd($G_k, \overline{G}_k) = 2k - 1$.

4 Sd($G, \overline{G}$) when $G$ is a tree or a unicyclic graph

In this section, we examine Sd($G, \overline{G}$) when $G$ is a tree or a unicyclic graph.

Trees

For any tree $T \neq P_4$, we show that Sd($T, \overline{T}$) = dim($T$) = max{dim($T$), dim($\overline{T}$)} and characterize trees $T$ satisfying Sd($T, \overline{T}$) = dim($T$) = dim($\overline{T}$). We also show that, for any tree $T$ that is not a path, dim($T$) $\geq$ dim($T$) $+$ ex($T$) $-$ 1.

We first consider Sd($T, \overline{T}$) for $T = P_n$, where $n \geq 2$. We recall the adjacency resolving set and adjacency dimension introduced by Jannesari and Omoomi in [9]. A set $W \subseteq V(G)$ is an adjacency resolving set of $G$ if, for any distinct vertices $x$ and $y$ in $G$, there exists a vertex $z \in W$ such that $z$ is adjacent to exactly one of the two vertices $x$ and $y$ in $G$. The adjacency dimension, adim($G$), of $G$ is the minimum of the cardinalities of all adjacency resolving sets of $G$.

Proposition 4.1. [9]
(a) For every graph $G$, adim($G$) = adim($\overline{G}$).
(b) If $G$ is a graph with dim($G$) = 2, then adim($G$) = dim($G$).

Proposition 4.2. [9] For $n \geq 4$, adim($C_n$) = adim($P_n$) = $\lceil \frac{2n+2}{5} \rceil$.

Proposition 4.3. For $n \geq 2$, Sd($P_n, \overline{P}_n$) = \begin{cases} 2 = \text{dim}(P_4) + \text{dim}(\overline{P}_4) & \text{if } n = 4, \\ \text{dim}(\overline{P}_n) & \text{otherwise}. \end{cases}

More explicitly, if $n \geq 2$, then Sd($P_n, \overline{P}_n$) = $\lceil \frac{2n+2}{5} \rceil$.

Proof. If $n \in \{2, 3\}$, then Sd($P_n, \overline{P}_n$) = 1 = dim($P_n$) = dim($\overline{P}_n$) by Theorem 2.7(a). If $n = 4$, then $P_4$ and $\overline{P}_4$ do not share a common end vertex. Thus, Sd($P_4, \overline{P}_4$) = 2 = dim($P_4$) + dim($\overline{P}_4$) by Theorem 2.6. So, suppose $n \geq 5$; then dim($P_n$) $\geq$ 4. Since $\overline{P}_n \not\cong K_n$, dim($\overline{P}_n$) = 2 by Proposition 2.9. Thus, Sd($P_n, \overline{P}_n$) = dim($\overline{P}_n$) = adim($\overline{P}_n$) = adim($P_n$) = $\lceil \frac{2n+2}{5} \rceil$ for $n \geq 5$, by Theorem 2.10, Proposition 4.1, and Proposition 4.2. \qed
Theorem 4.4. For any non-trivial tree $T \neq P_4$, $\text{Sd}(T, \overline{T}) = \dim(T)$.

Proof. Let $T$ be a tree of order $n \geq 2$ and let $T \neq P_4$. We consider four cases.

Case 1: $ex(T) = 0$. In this case, $T \cong P_n$ and $\text{Sd}(P_n, \overline{P_n}) = \dim(\overline{P_n})$, for $n \neq 4$, by Proposition 4.3.

Case 2: $ex(T) = 1$. Let $v$ be the exterior major vertex of $T$ with $\text{ter}_T(v) = k \geq 3$, and let $L_v(T) = \{\ell_1, \ldots, \ell_k\}$; further, let $d_T(v, \ell_1) \geq d_T(v, \ell_2) \geq \cdots \geq d_T(v, \ell_k)$ by relabeling the vertices of $T$ if necessary. If $d_T(v, \ell_1) = 1$, then $\text{diam}(T) = 2$ and $\text{Sd}(T, \overline{T}) = \dim(T) = n - 2 = \dim(T)$ by Theorem 2.10, Theorem 3.1 and Observation 2.1. If $d_T(v, \ell_1) = 2 = d_T(v, \ell_2)$, then $\text{diam}(T) = 3 = \dim(T)$ and $\text{Sd}(T, \overline{T}) = n - 3 = \dim(T) = \dim(T)$ by Proposition 3.3. If $d_T(v, \ell_1) = 2 = d_T(v, \ell_2)$ or $d_T(v, \ell_1) \geq 3$, then $\text{diam}(T) \geq 4$ and $\text{Sd}(T, \overline{T}) = \dim(T)$ by Theorem 2.10.

Case 3: $ex(T) = 2$. Let $v$ and $w$ be distinct exterior major vertices of $T$. If $V(T) - \{v, w\} = L_v(T) \cup L_w(T)$, then $\text{diam}(T) = 3 = \dim(T)$ and $\text{Sd}(T, \overline{T}) = n - 3 = \dim(T) = \dim(T) + 1$ by Proposition 3.3. If $V(T) - \{v, w\} \neq L_v(T) \cup L_w(T)$, then $\text{diam}(T) \geq 4$; thus, $\text{Sd}(T, \overline{T}) = \dim(T)$ by Theorem 2.10.

Case 4: $ex(T) \geq 3$. In this case, $\text{diam}(T) \geq 4$. By Theorem 2.10, $\text{Sd}(T, \overline{T}) = \dim(T)$.

Next, we characterize trees $T$ satisfying $\text{Sd}(T, \overline{T}) = \dim(T) = \dim(T)$. We begin with the following result.

Theorem 4.5. Every non-trivial tree $T$ satisfies the inequality $\dim(T) \geq \dim(T) + ex(T) - 1$, or equivalently,

$$\dim(T) \geq \sigma(T) - 1.$$  

(1)

Proof. The inequality (1) trivially holds for a path $T$.

Next, consider a tree $T$ with $ex(T) \geq 2$. Let $A$ denote the set of all paths leading from each $v \in M_1(T)$ to its terminal vertex. Let $B = \bigcup B_v$, where $B_v$ is the set of all paths leading from a major vertex $v \in M_2(T)$ to the terminal vertices associated with $v$. For each $v \in M_2(T)$, also let $B_v' = \{P - \{v\} : P \in B_v\}$. Assume, for contradiction, that there is a resolving set $S$ of $T$ with $|S| \leq \sigma(T) - 2$.

If there exists $v \in S \cap M_2(T)$, then $|S \cap V(T_v)| \geq \text{ter}_T(v)$. To see this, assume $|S \cap V(T_v)| < \text{ter}_T(v)$ and let $x$ and $y$ be two neighbors of $v$ in two paths of $B_v'$ omitted by $S$. Note $d_T(s, x) = 1 = d_{\overline{T}}(s, y)$ for $v \neq s \in S$. Since $ex(T) > 1$, there exists a vertex $z \notin N_T(v) \cup N_T(x) \cup N_T(y)$, and thus $z$ is adjacent to all three vertices $x, y, v$ in $T$. Since $x, y, v \notin N_T(v)$, $d_{\overline{T}}(x, v) > 1$ and $d_{\overline{T}}(y, v) > 1$. It follows that $d_{\overline{T}}(x, v) = 2 = d_{\overline{T}}(y, v)$. Therefore, $x$ and $y$ are not resolved by $S$ in $\overline{T}$.

Let the set of vertices $\{u, v, w\} \subseteq M(T)$ such that $u$ is distinct from $v$ and from $w$, whereas $v$ may equal $w$, be given. Note the observation (●) that no vertex of $T_u$ can resolve vertices $x$ and $y$ in $\overline{T}$ for $\{x, y\} \subseteq V(T_v) \cup V(T_w) - \{v, w\}$. Now, if $S$ contains a vertex $v_1$ with $\text{ter}_T(v_1) \geq 2$, we will pass from the triple $(S, B, A)$ to the triple $(S_1, B_1, A)$, where $S_1 = S - (S \cap V(T_{v_1}))$, with $|S_1| \leq \sigma(T) - 2 - \text{ter}_T(v_1)$, and $B_1 = B - B_{v_1}$. Through this “descent process”, we reach the triple $(S_0, B_0, A)$ where $|S_0| \leq |B_0| + |A| - 2$ and $S_0 \cap M_T(T) = \emptyset$.

Since $S_0$ omits at least two paths from collection $B_0 \cup A$, $S_0$ fails in $\overline{T}$ to resolve $u'$ from $v'$, neighbors of $u$ and $v$ along two omitted paths to their end vertices; note that $u$ and $v$ may denote the same major vertex. Now, by observation (●), the failure of $S_0$ to resolve $u'$ from $v'$ implies the failure of $S$ to do the same.

Finally, let a tree $T$ with $ex(T) = 1$ be given. Assume, to the contrary, that $\dim(T) < \sigma(T) - 1$. Let $v$ be the exterior major vertex of $T$ with $\text{ter}_T(v) = \alpha \geq 3$, and let $L_v(T) = \{\ell_1, \ldots, \ell_{\alpha}\}$. For each $i \in [\alpha]$, let $P_i^v$ denote the $v - \ell_i$ path excluding $v$ in $T$. If $v \notin S$, then $S \cap (V(P_i^v) \cup V(P_j^v)) = \emptyset$ for distinct $i, j \in [\alpha]$; thus, $\text{code}_{\overline{T}, S}(\ell_i) = \text{code}_{\overline{T}, S}(\ell_j)$, contradicting the assumption that $S$ is a
resolving set for $T$. So, suppose $v \in S$. Then $S \cap (V(P^a) \cup V(P^b) \cup V(P^c)) = \emptyset$ for three distinct $a, b, c \in \alpha$; let $d_T(v, \ell_a) \geq d_T(v, \ell_b) \geq d_T(v, \ell_c)$ by relabeling the vertices of $T$ if necessary. If $d_T(v, \ell_b) \geq 2$, then code$_{T,S}(\ell_b) = 1 = $ code$_{T,S}(\ell_b)$, where 1 is the all-one vector. If $d_T(v, \ell_b) = 1$, then code$_{T,S}(\ell_b) =$ code$_{T,S}(\ell_c)$, since $\ell_b$ and $\ell_c$ are twins in $T$ and $T$. In each case, $S$ fails to be a resolving set of $T$, contradicting the assumption. Thus, (1) holds for $ex(T) = 1$. 

\textbf{Theorem 4.6.} Let $T$ be a non-trivial tree. Then $Sd(T,T) = \dim(T) = \dim(T)$ if and only if $T$ satisfies one of the following:

(a) $T \in \{P_2, P_3\}$;

(b) $ex(T) = 1$ with $v \in M_2(T)$ such that $N_T(v) \cap L_v(T) \neq \emptyset$ and $d_T(v, \ell) \leq 2$ for each $\ell \in L_v(T)$.

\textbf{Proof.} Let $T$ be a tree of order $n \geq 2$. By Proposition 4.3 and Theorem 4.4, it suffices to characterize trees $T$ satisfying $Sd(T,T) = \dim(T)$.

\textbf{Proof.} If $T \in \{P_2, P_3\}$, then $Sd(T,T) = \dim(T) = 1$ by Theorem 2.7(a). If $T$ satisfies (b) of the current theorem such that $v \in M_2(T)$ and $d_T(v, \ell) = 1$, where $\ell \in L_v(T)$, then $N_T(v) - \{\ell\}$ forms a basis for both $T$ and $T$; thus, $Sd(T,T) = \dim(T)$.

\textbf{Proof.} If $ex(T) = 0$, then $T \cong P_2$, and $Sd(T, T) = \dim(P_2)$ implies $n \in \{2, 3\}$ by Theorem 2.5(a) and Theorem 2.7(a). If $ex(T) \geq 2$, then $Sd(T, T) = \dim(T) > \dim(T)$ by Theorem 4.4 and Theorem 4.5. So, suppose $ex(T) = 1$. Let $v$ be the exterior major vertex of $T$ with $ter_T(v) = k \geq 3$, and let $L_v(T) = \{\ell_1, \ldots, \ell_k\}$; further, let $d_T(v, \ell_1) \geq d_T(v, \ell_2) \geq \cdots \geq d_T(v, \ell_k)$ by relabeling the vertices of $T$ if necessary. For each $i \in [k]$, let $s_i \in N_T(v)$ such that $s_i$ lies on the $v - \ell_i$ path in $T$, and let $P_i$ denote the $s_i - \ell_i$ path in $T$. Since $Sd(T,T) = \dim(T)$ by the hypothesis, there exists some $j \in [k]$ with $S \cap V(P^j) = \emptyset$ by Theorem 3.2 and $d_T(v, \ell_j) = 1$; if $d_T(v, \ell_j) \geq 2$, then code$_{T,S}(s_j) = code_{T,S}(\ell_j)$, contradicting the assumption that $S$ is a resolving set for $T$. So, $N_T(v) \cap L_v(T) \neq \emptyset$ and $d_T(v, \ell_k) = 1$. We may assume, without loss of generality, that $|S \cap V(P^k)| = 1$ for each $i \in [k - 1]$ and $S \cap V(P^k) = \emptyset$. It suffices to show that $d_T(v, \ell_k) \leq 2$. Assume, to the contrary, that $d_T(v, \ell_k) = d \geq 3$ and let $v - \ell_k$ path be given by $v = t_0, s_1 = t_1, t_2, \ldots, t_d = \ell_1$. If $t_1 \in S$, then code$_{T,S}(\ell_1) = code_{T,S}(\ell_k)$. If $t_2 \in S$, then code$_{T,S}(t_1) = code_{T,S}(t_3)$. If $S \cap \cup_{i=3}^d \{t_i\} = \emptyset$, then code$_{T,S}(t_1) = code_{T,S}(\ell_k)$. So, $d_T(v, \ell_k) \leq 2$. 

\textbf{Unicyclic graphs}

For any unicyclic graph $G$ of order $n \geq 3$, we show that $Sd(G, G) \in \{\dim(G), 1 + \dim(G)\}$; moreover, we show that $Sd(G, G) = \dim(G)$ for $n \geq 7$.

First, we determine $Sd(C_n, \overline{C}_n)$ for $n \geq 3$. It is well known that $\dim(C_n) = 2$, where $n \geq 3$.

\textbf{Proposition 4.7.} For $n \geq 3$,

$$Sd(C_n, \overline{C}_n) = \left\{ \begin{array}{ll} 2 & \text{if } n = 3, \\ \left\lceil \frac{2n+2}{3} \right\rceil & \text{if } n \geq 4. \end{array} \right.$$ 

\textbf{Proof.} If $n = 3$, then $Sd(C_3, \overline{C}_3) = 2$ by Theorem 2.7(b). If $n \in \{4, 5\}$, then $\dim(C_n) = 2$; thus, $Sd(C_n, \overline{C}_n) = \dim(C_n) = 2$ by Theorem 2.10. If $n \geq 6$, then $\dim(C_n) = 2$, and thus $\dim(C_n, \overline{C}_n) = \dim(C_n) = adim(C_n) = adim(C_n) = \left\lceil \frac{2n+2}{3} \right\rceil$ by Theorem 2.10, Proposition 4.1 and Proposition 4.2.

Next, we consider $Sd(G, \overline{G})$ for an arbitrary unicyclic graph $G$.

\textbf{Lemma 4.8.} Let $G$ be a unicyclic graph of order $n \geq 4$ with $\dim(G) = 2$. Then $G \in \{C_4, C_5\}$, or $G$ is isomorphic to the graph $H$ obtained from $K_{1,n-1}$ by joining two end vertices by an edge. Moreover, 

$$Sd(G, \overline{G}) = \dim(G) = \left\{ \begin{array}{ll} \dim(G) & \text{if } G \in \{C_4, C_5\} \text{ or } G = H \text{ with } n \geq 5, \\ \dim(G) + 1 & \text{if } G = H \text{ with } n = 4. \end{array} \right.$$
Proof. Let $G$ be a unicyclic graph of order $n \geq 4$ with $\text{diam}(G) = 2$, and let $C = C_m$ be the unique cycle of a unicyclic graph $G$ given by $u_1, u_2, \ldots, u_m, u_1$, where $m \geq 3$. Then $m \in \{3, 4, 5\}$, since $m \geq 6$ implies $\text{diam}(G) \geq 3$.

First, suppose $m = 3$. Since $\text{diam}(G) = 2$, $C$ must contain exactly one major vertex with all its terminal vertices adjacent to it (note that $G$ is isomorphic to $H$); let $\ell_1, \ell_2, \ldots, \ell_{n-3}$ be the terminal vertices of its major vertex, say $u_1$, in $G$ with $d_G(u_1, \ell_i) = 1$, where $i \in [n-3]$. If $n = 4$, then $\{u_2, u_3\}$ forms a basis for $G$, whereas $\{u_2, u_3\}$ forms a basis for $G$; thus $\text{Sd}(G, \overline{G}) = \text{dim}(G) = 2 + \text{dim}(\overline{G})$.

If $n \geq 5$, then $\text{Sd}(G, \overline{G}) = \text{dim}(G) = n - 3 = \text{dim}(\overline{G})$: (i) for any resolving set $S$ of either $G$ or $\overline{G}$, $|S \cap \{u_2, u_3\}| \geq 1$ and $|S \cap (\cup_{i=1}^{n-3} \{\ell_i\})| \geq n - 4$ by Observation 2.1; (ii) $\{u_2 \cup (\cup_{i=1}^{n-3} \{\ell_i\})\}$ forms a resolving set for both $G$ and $\overline{G}$.

Next, suppose $m \in \{4, 5\}$. Since $\text{diam}(G) = 2$, $G = C$; thus $G \in \{C_4, C_5\}$. Since any two adjacent vertices of $G$ form a resolving set for both $G$ and $\overline{G}$, $\text{Sd}(G, \overline{G}) = \text{dim}(G) = 2 = \text{dim}(\overline{G})$. □

**Proposition 4.9.** Let $G$ be a unicyclic graph of order at least three. Let $H_1$ be the graph obtained from $K_{1,3}$ by joining two end vertices by an edge, and let $H_2$ be the graph obtained from $P_5$ by adding an edge between the two support vertices. Then

$$\text{Sd}(G, \overline{G}) = \begin{cases} \text{dim}(G) = \text{dim}(\overline{G}) + 1 = 2 & \text{if } G = H_1, \\ \text{dim}(G) + 1 = \text{dim}(\overline{G}) + 1 & \text{if } G = H_2, \\ \text{dim}(\overline{G}) & \text{otherwise.} \end{cases}$$

Proof. Let $G$ be a unicyclic graph. We consider three cases as follows: (i) $\text{diam}(G) = 1$; (ii) $\text{diam}(G) = 2$ or $\text{diam}(\overline{G}) = 2$; and (iii) $\text{diam}(G) = 3 = \text{diam}(\overline{G})$.

First, suppose $\text{diam}(G) = 1$; then $G = C_3$ and $\text{Sd}(C_3, \overline{C}_3) = 2 = \text{dim}(\overline{C}_3) = \text{dim}(C_3)$ by Theorem 2.7(b).

Second, suppose $\text{diam}(G) = 2$ or $\text{diam}(\overline{G}) = 2$. If $\text{diam}(G) = 2$, then $\text{Sd}(H_1, \overline{H}_1) = 2 = \text{dim}(H_1) + 1 = \text{dim}(\overline{H}_1)$ and $\text{Sd}(G, \overline{G}) = \text{dim}(\overline{G})$ if $G \neq H$ by Lemma 4.8. If $\text{diam}(\overline{G}) = 2$, then $\text{Sd}(G, \overline{G}) = \text{dim}(\overline{G})$ by Theorem 2.10.

Third, suppose $\text{diam}(G) = 3 = \text{diam}(\overline{G})$. Then $\text{Sd}(H_2, \overline{H}_2) = \text{dim}(\overline{H}_2) + 1 = \text{dim}(H_2) + 1$ and $\text{Sd}(G, \overline{G}) = \text{dim}(\overline{G})$ for $G \neq H_2$ by Proposition 3.5. □

## 5 Some remarks and open problems

In this section, we provide a realization result for $\text{Sd}(G, \overline{G})$. In view of Corollary 2.4, we provide examples showing that $\min\{\text{dim}(G) + \text{dim}(\overline{G}), |V(G)| - 1\} - \text{Sd}(G, \overline{G})$ can be arbitrarily large. We conclude this paper with some open problems.

**Theorem 5.1.** For integers $n, k$ with $3 \leq k \leq n - 2$, there exists a connected graph $G$ of order $n$ with $\text{Sd}(G, \overline{G}) = k$.

Proof. Let a pair of integers $(n, k)$ with $n \geq 5$ and $3 \leq k \leq n - 2$ be given. If $G$ is the tree in Figure 4(a), then all but one vertex in $N_G(v)$ forms a basis for both $G$ and $\overline{G}$, and thus $k = n - 2$.

Now, suppose $G$ is the tree in Figure 4(b) with $\text{ex}(T) = 2$, $\text{ter}_G(v_1) = a \geq 2$ and $\text{ter}_G(v_2) = b \geq 2$, where $0 \leq x \leq a - 1$ and $0 \leq y \leq b - 1$. If $x = 1$, let $S_1 = N_G(v_1) \cup N_G(v_2) - \{v_2, \ell_a, m_b\}$; if $x \neq 1$, let $S_2 = N_G(v_1) \cup N_G(v_2) - \{v_1, \ell_b, m_a\}$. Note that $|S_1| = |S_2| = \sigma(G) - 1$. Then $k = \text{dim}(\overline{G}) = \sigma(G) - 1$, where $3 \leq k = n - (x + y + 3) \leq n - 3$, by Theorem 4.4, Theorem 4.5, and the fact that $S_1$ (when $x = 1$) or $S_2$ (when $x \neq 1$) is a resolving set for $\overline{G}$.

**Proposition 5.2.** There exists a graph $G$ such that $\min\{\text{dim}(G) + \text{dim}(\overline{G}), |V(G)| - 1\} - \text{Sd}(G, \overline{G})$ can be arbitrarily large.

Proof. Let $G$ be the tree in Figure 5 with $k \geq 2$ exterior major vertices $v_1, \ldots, v_k$ such that $\text{ter}_G(v_i) = 2$ for each $i \in [k]$. Then $\text{dim}(G) = k$ by Theorem 3.1 and $\text{Sd}(G, \overline{G}) = \text{dim}(\overline{G}) = 2k - 1$ by
Theorem 4.4, Theorem 4.5 and the fact that \((\cup_{k=1}^{n-1} L_v(G)) \cup \{\ell_k\}\) forms a resolving set for \(G\). So, \(\dim(G) + \dim(\overline{G}) = 3k - 1 = |V(G)| - 1\) and \(\min\{\dim(G) + \dim(\overline{G}), |V(G)| - 1\} - \text{Sd}(G, \overline{G}) = 3k - 1 - (2k - 1) = k\) can be arbitrarily large, as \(k \to \infty\).

Figure 5: A graph \(G\) such that \(\min\{\dim(G) + \dim(\overline{G}), |V(G)| - 1\} - \text{Sd}(G, \overline{G})\) can be arbitrarily large, where \(k \geq 2\).

We conclude this paper with some open problems.

**Question 1.** Let \(G\) and \(\overline{G}\) be connected graphs of order \(n \geq 4\). Then \(2 \leq \text{Sd}(G, \overline{G}) \leq n - 2\) by Corollary 2.4 and Theorem 2.7. Can we characterize graphs \(G\) satisfying \(\text{Sd}(G, \overline{G})\) equals 2 and \(n - 2\), respectively?

**Question 2.** Note that \(\text{Sd}(P_4, \overline{P}_4) = \dim(P_4) + \dim(\overline{P}_4)\). Is there any other graph \(G\) such that \(\text{Sd}(G, \overline{G}) = \dim(G) + \dim(\overline{G})\)?

**Question 3.** As shown in Theorem 3.6, the self-complementary split graph \(G_k\) with \(\text{diam}(G_k) = 3\) satisfies \(\text{Sd}(G_k, \overline{G_k}) > \max\{\dim(G_k), \dim(\overline{G_k})\}\), where \(k\) is a positive integer. Is there a graph \(G\) such that \(\text{Sd}(G, \overline{G}) - \max\{\dim(G), \dim(\overline{G})\}\) can be arbitrarily large? Conjecture 3.7 would imply \(\text{Sd}(G_k, \overline{G_k}) - \max\{\dim(G_k), \dim(\overline{G_k})\} \to \infty\) as \(k \to \infty\).

**Question 4.** If \(\{\text{diam}(G), \text{diam}(\overline{G})\} \neq \{3\}\), then \(\text{Sd}(G, \overline{G}) = \max\{\dim(G), \dim(\overline{G})\}\) by Theorem 2.10. Can we characterize graphs \(G\) such that \(\text{diam}(G) = 3 = \text{diam}(\overline{G})\) and \(\text{Sd}(G, \overline{G}) = \max\{\dim(G), \dim(\overline{G})\}\)?

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