Total Roman domination in direct product graphs*

Abel Cabrera Martínez(1), Dorota Kuziak(2), Iztok Peterin(3,4) and Ismael G. Yero(5)

(1) Departament d’Enginyeria Informàtica i Matemàtiques
Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain
abel.cabrera@urv.cat

(2) Departamento de Estadística e Investigación Operativa, Escuela Politécnica Superior de Algeciras
Universidad de Cádiz, Av. Ramón Puyol s/n, 11202 Algeciras, Spain.
dorota.kuziak@uca.es

(3) Faculty of Electrical Engineering and Computer Science
University of Maribor, Koroška cesta 46, 2000 Maribor, Slovenia.
iztok.peterin@um.si

(4) Institute of Mathematics, Physics and Mechanics
Jadranska ulica 19, 1000 Ljubljana, Slovenia.

(5) Departamento de Matemáticas, Escuela Politécnica Superior de Algeciras
Universidad de Cádiz, Av. Ramón Puyol s/n, 11202 Algeciras, Spain.
ismael.gonzalez@uca.es

Abstract

Given a graph $G$, a total Roman dominating function for $G$ is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex with label 0 is adjacent to a vertex with label 2, and the set of vertices with positive labels induces a graph of minimum degree at least one. The total Roman domination number $\gamma_{tR}(G)$ of $G$ is the smallest possible value of $\sum_{v \in V(G)} f(v)$ among all total Roman dominating functions $f$. The total Roman domination number of the direct product graph $G \times H$ of the graphs $G$ and $H$ is studied in this work. Specifically, several relationships, in the shape of upper and lower bounds, between $\gamma_{tR}(G \times H)$ and some classical domination parameters for the factors are given. Characterizations of the direct product graphs $G \times H$ achieving small values ($\leq 7$) for $\gamma_{tR}(G \times H)$ are presented, and exact values for $\gamma_{tR}(G \times H)$ are deduced, while considering various specific direct product classes.

Keywords: Total Roman domination; direct product graphs.

AMS Subject Classification Numbers: 05C69, 05C76.

*The third author was partially supported by Slovenian research agency under the grants P1-0297, J1-1693 and J1-9109.
1 Introduction

This work is aimed to present a study of the total Roman domination number of direct product graphs. Studies concerning domination related parameters in graphs are very frequently present in the last recent years. This might probably be caused by the popularity of some classical problems, like for instance Vizing’s conjecture [16, 17], which states that the domination number of Cartesian product of two graphs is not smaller than the product of the domination numbers of the factors of the product. See [3], for a survey and recent results concerning this conjecture. Several other problems concerning domination parameters in product graphs have attracted the attention of a large number of researchers. Works of that type concerning direct product graphs are [4, 7, 11, 13].

The (total) Roman domination variants are among the most popular topics of domination in graphs. Both versions have had their birth in connection with some defense strategies related to the ancient Roman Empire (see [12, 14]). Studies on (total) Roman domination in product graphs have not escaped from the researchers attention. For instance, [5, 15, 18] are aimed to these goals, although no works appear that considers the Roman domination parameters for the case of direct products. We continue with contributions to the study of domination related parameters in product graphs, specifically we center our attention on the total Roman domination number of direct products.

In this work, we consider simple graphs without isolated vertices. For a function \( f : V(G) \rightarrow \{0, 1, 2\} \) and a set of vertices \( S \subseteq V(G) \), the weight of \( S \) under \( f \) is \( f(S) = \sum_{v \in S} f(v) \). Moreover, the weight of \( f \) is \( \omega(f) = f(V(G)) \). Since the function \( f \) generates three sets \( V_0, V_1, V_2 \) such that \( V_i = \{v \in V(G) : f(v) = i\}, i \in \{0, 1, 2\} \), we shall write \( f = (V_0, V_1, V_2) \).

A function \( f = (V_0, V_1, V_2) \) is called a Roman dominating function on \( G \) if every vertex \( v \in V_0 \) is adjacent to a vertex \( u \in V_2 \). The Roman domination number of \( G \) is the minimum possible weight among all Roman dominating functions on \( G \), and is denoted by \( \gamma_R(G) \). Roman domination concepts in graph were formally introduced in [6], motivated in part by some historical roots coming from the ancient Roman Empire (see for instance [12, 14]). A Roman dominating function \( f = (V_0, V_1, V_2) \) is called a total Roman dominating function if the set \( V_1 \cup V_2 \) induces a graph without isolated vertices. The total Roman domination number of \( G \), denoted by \( \gamma_{tR}(G) \), is the minimum possible weight among all total Roman dominating functions on \( G \). A total Roman dominating function of weight \( \gamma_{tR}(G) \) is called a \( \gamma_{tR}(G) \)-function. Total Roman domination concepts were first introduced in [10] by using some more general settings. The concepts were further specifically introduced and firstly well studied in [2].

A set \( D = \{v_1, \ldots, v_r\} \subseteq V(G) \) is called a packing of \( G \), if \( N[v_i] \cap N[v_j] = \emptyset \) for any distinct \( i, j \in \{1, \ldots, r\} \). The packing number of \( G \) is the largest possible cardinality among all packing sets of \( G \), and is denoted by \( \rho(G) \). A packing set induces a subgraph of maximum degree 0, i.e., a graph without edges. If we substitute the closed neighborhoods with open neighborhood in the definition above, then the concept of an open packing set arises. That is, a set \( D \) is an open packing set, if \( N(v_i) \cap N(v_j) = \emptyset \) for any distinct \( i, j \in \{1, \ldots, r\} \). Similarly, the open packing number of \( G \), denoted by \( \rho_o(G) \), is the largest possible cardinality among all open packing sets of \( G \). We remark that an open packing set is a set of vertices of the graph which induces a graph of maximum degree 1, but that could have vertices of degree 0.

A set \( D \subseteq V(G) \) is a total dominating set if every vertex from \( V(G) \) is adjacent to a vertex from \( D \). The minimum cardinality among all total dominating sets of \( G \) is called the total domination...
number of $G$ and is denoted by $\gamma_t(G)$. A total dominating set of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. If there exists a total dominating set of $G$ that is at the same time also an open packing, then $G$ is called an efficient open domination graph.

The direct product of graphs $G$ and $H$ is the graph $G \times H$ with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{(g, h)(g', h') : gg' \in E(G), hh' \in E(H)\}$. In Figure 1 we show the graph $P_6 \times P_6$. As usual, we call the map $p_G : (g, h) \mapsto g$ a projection of $G \times H$ onto $G$ and the map $p_H : (g, h) \mapsto h$ a projection of $G \times H$ onto $H$. Set $G^h = \{(g, h) : g \in V(G)\}$ is called a $G$-layer through $h \in V(H)$ and contains all vertices that project to $h$. Similar is defined an $H$-layer $H^g = \{(g, h) : h \in V(H)\}$ through $g \in V(G)$. Vertices from $G^h$ and $H^g$ form an independent set of $G \times H$.

The direct product is the graph product (see the exhausting monograph on graph products [8]) in categorical sense as the end vertices of every edge from $G \times H$ project to end vertices of edges in both factors. This makes the direct product the most natural among all graph products, but, on the other hand, this also makes this product the most elusive one in many perspectives. So, the connectedness of both factors $G$ and $H$ does not imply the connectedness of the product $G \times H$. (Notice that $P_6 \times P_6$ from Figure 1 is not connected.) To achieve this, one of the factors must also be non-bipartite, see Theorem 5.9 in [8]. One reason for this is that layers form independent sets in $G \times H$. On the other side, the open neighborhoods behave nice in the direct product with respect to the factors as

$$N_{G \times H}(g, h) = N_G(g) \times N_H(h). \quad (1)$$

Two different total Roman dominating functions on $P_6 \times P_6$ are presented on Figure 1.

![Figure 1: Two total Roman dominating functions on $P_6 \times P_6$ where vertices in $V_0$ are white circles, vertices in $V_1$ are black circles and black squares represent vertices in $V_2$.](image)

The degree $\delta_G(v)$ of a vertex in a graph $G$ is the cardinality of its open neighborhood, that is $\delta_G(v) = |N_G(v)|$. The maximum degree of a vertex in a graph $G$ is denoted by $\Delta(G)$. Clearly, $1 \leq \Delta(G) \leq |V(G)| - 1$ as we consider only simple graphs without isolated vertices. A vertex $v \in V(G)$ with $\delta_G(v) = 1$ is called a leaf and if $\delta_G(v) = |V(G)| - 1$, then $v$ is a universal vertex of $G$. In the direct product $G \times H$ we have $\delta_{G \times H}(g, h) = \delta_G(g)\delta_H(h)$ and $\Delta(G \times H) = \Delta(G)\Delta(H)$ by (1).
2 General bounds

We start our exposition with general upper and lower bounds for $\gamma_{tR}(G \times H)$ which are depending on $\rho(G)$, $\rho(H)$, $\gamma_{tR}(G)$ and $\gamma_{tR}(H)$.

Theorem 1. If $g = (A_0, A_1, A_2)$ is a $\gamma_{tR}(G)$-function (with maximum cardinality of $A_2$) and $h = (B_0, B_1, B_2)$ is a $\gamma_{tR}(H)$-function (with maximum cardinality of $B_2$), then

$$\max\{\rho(G)\gamma_{tR}(H), \rho(H)\gamma_{tR}(G)\} \leq \gamma_{tR}(G \times H) \leq \gamma_{tR}(G)\gamma_{tR}(H) - 2|A_2||B_2|.$$ 

Proof. We consider a function $f$ on $G \times H$ defined as follows. If $(u, v) \in (A_2 \times (B_1 \cup B_2)) \cup (A_1 \times B_2)$, then $f(u, v) = 2$; if $(u, v) \in (A_1 \times B_1)$, then $f(u, v) = 1$; and $f(u, v) = 0$ otherwise. If $f(u, v) \geq 1$, then since $g(u) \geq 1$ and $h(v) \geq 1$, there exist two vertices $u' \in N_G(u)$ and $v' \in N_H(v)$ such that $g(u') \geq 1$ and $h(v') \geq 1$. Thus, it follows $(u', v') \in N_G(u, v)$ and $f(u', v') \geq 1$. Now, consider a vertex $(u, v) \in V(G \times H)$ such that $f(u, v) = 0$. If $(u, v) \in A_0 \times V(H)$, then there exist two vertices $u'' \in N_G(u)$ and $v'' \in N_H(v)$ such that $g(u'') = 2$ and $h(v'') \geq 1$. Thus, it follows $(u'', v'') \in N_{G \times H}(u, v)$ and $f(u'', v'') = 2$. Finally, if $(u, v) \in A_1 \times B_0$ with $i \in \{1, 2\}$, then a symmetrical argument to the above one produce a similar conclusion.

As a consequence, it follows that $f$ is a total Roman dominating function on $G \times H$, and we obtain that

$$\gamma_{tR}(G \times H) \leq \omega(f)$$

$$= 2|A_2 \times B_2| + 2|A_2 \times B_1| + 2|A_1 \times B_2| + |A_1 \times B_1|$$

$$= (2|A_2| + |A_1|)(|B_2| + |B_1|) + |A_1||B_2|$$

$$= (2|A_2| + |A_1|)(2|B_2| + |B_1|) - 2|A_2||B_2|$$

$$= \gamma_{tR}(G)\gamma_{tR}(H) - 2|A_2||B_2|.$$ 

In order to obtain the lower bound, we consider a $\gamma_{tR}(G \times H)$-function $f$ and a $\rho(G)$-set $S = \{u_1, \ldots, u_{\rho(G)}\}$. Now, for every $i \in \{1, \ldots, \rho(G)\}$, we construct a function $h_i$ on $H$ as follows. For every $v \in V(H)$, $h_i(v) = \max\{f(u, v) : u \in N_G[u_i]\}$.

If $h_i(v) \geq 1$, then there exists a vertex $(u, v) \in N_G[u_i] \times \{v\}$ for which $f(u, v) \geq 1$. If $f(u, v) = 0$, then there exists a vertex $(x, y) \in N_G(u_i) \times N_H(v)$ such that $f(x, y) = 2$ and $(x, y) \in N_{G \times H}(u_i, v)$. Moreover, note that in this case $h_i(y) = 2$ and also that $y \in N_H(v)$. Now, if $f(u, v) \geq 1$, then there exists a vertex $(x', y') \in N_{G \times H}(u_i, v)$ such that $f(x', y') \geq 1$. In such situation, we similarly get $h_i(y') \geq 1$ and $y' \in N_H(v)$.

On the other hand, if $h_i(v) = 0$, then for every vertex $(u, v) \in N_G[u_i] \times \{v\}$ we have $f(u, v) = 0$. Particularly, for the vertex $(u_i, v)$, there exists a vertex $(u_i', v) \in N_{G \times H}(u_i, v)$ with $v' \neq v$ and $f(u_i', v') = 2$. Thus, the vertex $v' \in V(H)$ satisfies that $v' \in N_H(v)$ and $h_i(v') = 2$.

As a consequence of these arguments, we deduce that $h_i$ is a total Roman dominating function on $H$ whose weight is less than or equal to $f(N_G[u_i] \times V(H))$, i.e., $\gamma_{tR}(H) \leq f(N_G[u_i] \times V(H))$. Hence, we have the following.

$$\gamma_{tR}(G \times H) \geq \sum_{i=1}^{\rho(G)} f(N_G[u_i] \times V(H)) \geq \sum_{i=1}^{\rho(G)} \gamma_{tR}(H) = \rho(G)\gamma_{tR}(H).$$
By the symmetry of the product, we also deduce that $\gamma_{tR}(G \times H) \geq \rho(H)\gamma_{tR}(G)$, which completes the proof of the lower bound.

Since every graph of order at least three contains at least one total Roman dominating function of minimum weight having at least one vertex labeled two, the next result is a direct consequence of the theorem above.

**Corollary 2.** For any graphs $G$ and $H$ without isolated vertices of orders at least three,

$$\gamma_{tR}(G \times H) \leq \gamma_{tR}(G)\gamma_{tR}(H) - 2.$$  

Notice that we can avoid the remarks about maximum cardinality of $A_2$ and $B_2$ in Theorem 1. However, the bound is better if we take a $\gamma_{tR}(G)$-function and a $\gamma_{tR}(H)$-function with maximum cardinality of $A_2$ and $B_2$, respectively. The proof of the upper bound from Theorem 1 remains valid for any total Roman dominating functions $g$ and $h$ of graphs $G$ and $H$ without isolated vertices, respectively, as long as we exchange $\gamma_{tR}(G)$ and $\gamma_{tR}(H)$ by $\omega(g)$ and $\omega(h)$, respectively, in the last step of the proof. Therefore the upper bound of Theorem 1 can be improved as follows.

**Remark 3.** For any two graphs $G$ and $H$ without isolated vertices,

$$\gamma_{tR}(G \times H) \leq \min\{\omega(g)\omega(h) - 2|A_2||B_2|\},$$

where the minimum is taken over all total Roman dominating functions $g = (A_0, A_1, A_2)$ and $h = (B_0, B_1, B_2)$ on $G$ and $H$, respectively.

Despite the fact that the bound above represents an improvement of the upper bound of Theorem 1, we are not aware of any pair of graphs $G$ and $H$ where the upper bound of Remark 3 is better than the upper bound of Theorem 1.

Let $D_G$ be a $\gamma_t(G)$-set. Clearly, $g = (V(G) - D_G, \emptyset, D_G)$ is a total Roman dominating function on $G$ and $\omega(g) = 2\gamma_t(G)$. Remark 3 yields the following connection.

**Corollary 4.** For any graphs $G$ and $H$ without isolated vertices,

$$\gamma_{tR}(G \times H) \leq 2\gamma_t(G)\gamma_t(H).$$

If $G$ and $H$ are efficient open domination graphs, then $\rho_o(G) = \gamma_t(G)$ and $\rho_o(H) = \gamma_t(H)$ (see Observation 1.1 from [9]) and Corollary 4 implies the following.

**Corollary 5.** If $G$ and $H$ are efficient open domination graphs, then $\gamma_{tR}(G \times H) \leq 2\rho_o(G)\rho_o(H)$.

A graph $G$ is known to be a total Roman graph if it satisfies that $\gamma_{tR}(G) = 2\gamma_t(G)$. In the case of two total Roman graphs we can develop the upper bound of Corollary 4 to the following result.

**Corollary 6.** If $G$ and $H$ are two total Roman graphs, then

$$\gamma_{tR}(G \times H) \leq \frac{\gamma_{tR}(G)\gamma_{tR}(H)}{2}.$$
The lower bound of Theorem 1 can be improved by factor 2 if one factor is bipartite and the other without triangles as shown next.

**Theorem 7.** If \( G \) is a triangle free graph and \( H \) is a bipartite graph of order at least two without isolated vertices, then

\[
\gamma_{tR}(G \times H) \geq 2 \rho(G) \gamma_{tR}(H).
\]

**Proof.** Let \( f \) and \( S \) be defined as in the proof of the lower bound of Theorem 1. Clearly, for any vertex \( u_i \in S \), \( N_G[u_i] \times V(H) \) induces a non connected graph with at least two components. In this sense, for every \( i \in \{1, \ldots, \rho(G)\} \) and for every component of the subgraph induced by \( N_G[u_i] \times V(H) \), we can construct a total Roman dominating function in the same style as in the proof of Theorem 1. This means that \( f(N_G[u_i] \times V(H)) \geq 2 \gamma_{tR}(H) \). A similar arguments as the one in the proof of Theorem 1 leads to the desired bound.

By using a similar argument as the one in the proof of the lower bound of Theorem 1, but using an open packing instead of a packing, we can also deduce the lower bound in the following result.

**Theorem 8.** For any graphs \( G \) and \( H \) without isolated vertices of orders at least three,

\[
\gamma_{tR}(G \times H) \geq \max \left\{ \frac{\rho_o(H) \gamma_{tR}(G)}{2}, \frac{\rho_o(G) \gamma_{tR}(H)}{2} \right\}.
\]

**Proof.** The lower bound follows by considering a similar partition of the vertex of \( G \), as the one used while proving the lower bound of Theorem 1, but instead of using only one vertex as the “center” of each set of the partition, we might need to use now two adjacent vertices as the “centers”. This is based on the structure of open packing sets.

We consider a \( \gamma_{tR}(G \times H) \)-function \( f \), and a \( \rho_o(G) \)-set \( S = S_0 \cup S_1 \) such that \( S_0 \) induces a graph without edges and \( S_1 \) induces a regular graph of degree 1. Note that \( S_0 \) or \( S_1 \) could be empty (although not both at the same time). Now, for every \( u_i \in S_0 \), we construct a function \( h_i(v) = \max\{f(u,v) : u \in N_G[u_i]\} \) for every \( v \in V(H) \).

In the same manner, as in the proof of the lower bound of Theorem 1, we deduce that \( h_i \) is a total Roman dominating function on \( H \), and so, \( \gamma_{tR}(H) \leq f(N_G[u_i] \times V(H)) \) for every \( u_i \in S_0 \).

Now, for any pair of adjacent vertices \( w_i, w'_i \in S_1 \), we construct a function \( h'_i \) on \( H \) as follows. For every \( v \in V(H) \), \( h'_i(v) = \max\{f(w,v) : w \in N_G(w_i) \cup N_G(w'_i)\} \). From now on, let \( N_i = N_G(w_i) \cup N_G(w'_i) \) and note that \( w_i, w'_i \in N_i \).

If \( h'_i(v) \geq 1 \), then there exists a vertex \( (w,v) \in N_i \times \{v\} \) for which \( f(w,v) \geq 1 \). Assume (w.l.o.g.) that \( w \) is adjacent to \( w_i \) in \( G \) (note that \( w \) could be \( w'_i \)). If \( f(w_i,v) = 0 \), then there exists a vertex \( (x,y) \in N_G(w_i) \times N_H(v) \) such that \( f(x,y) = 2 \) and \( (x,y) \in N_{G \times H}(w_i,v) \). Also, \( h'_i(y) = 2 \) and \( y \in N_H(v) \). If \( f(w_i,v) \geq 1 \), then there exists a vertex \( (x',y') \in N_{G \times H}(w_i,v) \) such that \( f(x',y') \geq 1 \). In such situation, we get \( h'_i(y') \geq 1 \) and \( y' \in N_H(v) \) as well.

Now, if \( h'_i(v) = 0 \), then for every vertex \( (w,v) \in N_i \times \{v\} \) we have \( f(w,v) = 0 \). Particularly, for the vertex \( (w_i,v) \) (or for \( (w'_i,v) \) as well), there exists a vertex \( (z,v') \in N_{G \times H}(w_i,v) \) with \( v' \neq v \) and \( f(z,v') = 2 \). Thus, for the vertex \( v' \in V(H) \) we have \( v' \in N_H(v) \) and \( h'_i(v') = 2 \).

As a consequence of these arguments, we deduce that \( h'_i \) is a total Roman dominating function on \( H \) whose weight is less than or equal to \( f(N_i \times V(H)) = f((N_G(w_i) \cup N_G(w'_i)) \times V(H)) \), i.e.,
Theorem 10. The following assertions holds for graphs $G$ and $H$ without isolated vertices.

3 Direct product graphs with small $\gamma_{tR}(G \times H)$

We concentrate our attention in this section on the case when $\gamma_{tR}(G \times H)$ is small. We shall characterize all the direct products graphs $G \times H$ for which $\gamma_{tR}(G \times H) \leq 7$. For this we need the following class of graphs.

A graph $G$ is called triangle centered if there exists a triangle $C_3 = xyz$ in $G$ such that every vertex of $G$ is adjacent to at least two vertices of $C_3$. We call such $C_3$ the central triangle of a triangle centered graph. Notice that any two vertices of a central triangle form a total dominating set of a triangle centered graph $G$ and we have $\gamma_t(G) = 2$.

Theorem 10. The following assertions holds for graphs $G$ and $H$ without isolated vertices.
(i) There are no graphs $G$ and $H$ for which $\gamma_{tR}(G \times H) \in \{1, 2, 3, 5\}$.

(ii) $\gamma_{tR}(G \times H) = 4$ if and only if $G \cong H \cong K_2$.

(iii) $\gamma_{tR}(G \times H) = 6$ if and only if $(G$ and $H$ have at least two universal vertices each and at least one of them is of order at least three), or $(G$ and $H$ have a universal vertex, one of the graph $G$ and $H$ exactly one universal vertex, and the other one is different from $K_2$, and only one of $G$ and $H$ can be triangle centered).

(iv) $\gamma_{tR}(G \times H) = 7$ if and only if both $G$ and $H$ have a universal vertex, one of the graph $G$ and $H$ has exactly one universal vertex, and the other one is different from $K_2$, and only one of $G$ and $H$ can be triangle centered.

(v) If at most one of the graphs $G$ and $H$ has a universal vertex, $\gamma_t(G) = \gamma_t(H) = 2$, and $G$ and $H$ are not both triangle centered, then $\gamma_{tR}(G \times H) = 8$.

Proof. For (i) notice that there must be at least two adjacent vertices $(g, h)$ and $(g', h')$ in $V_1 \cup V_2$ for a $\gamma_{tR}(G \times H)$-function $f = (V_0, V_1, V_2)$. If $|V_1 \cup V_2| = 2$, then $(g, h')$ and $(g', h)$ have label 0 and no neighbor with label 2, a contradiction. This already shows that $\gamma_{tR}(G \times H) \geq 3$. If $\gamma_{tR}(G \times H) = 3$, then either $|V_1 \cup V_2| = 2$, which is not possible, or $|V_1 \cup V_2| = 3$. In later case there are three vertices of label 1 and no vertex of label 2, a contradiction as we have $|V(G \times H)| \geq 4$. Hence $\gamma_{tR}(G \times H) > 3$.

To end with (i) suppose that $\gamma_{tR}(G \times H) = 5$. Let first $|V_2| = 2$ where $(g, h), (g_1, h_1) \in V_2$. If $g \neq g_1$ and $h \neq h_1$, then only one vertex from $(g, h_1)$ and $(g_1, h)$ can have label 1 and the other has label 0 and is not adjacent to a vertex of label 2, a contradiction. So, either $g = g_1$ or $h = h_1$, say $g = g_1$. In $V_1$ is only one vertex, say $(g_2, h_2)$, and it must be adjacent to both vertices of $V_2$. This means that $h_2 \neq h$ and $h_2 \neq h_1$. But then $(g, h_2)$ posses label 0 and is not adjacent to a vertex of label 2, a contradiction.

So let $|V_2| = 1$ where $(g, h) \in V_2$ and $(g', h') \in V_1$ is adjacent to $(g, h)$. There are only two more vertices in $V_1$ and these vertices must be $(g, h')$ and $(g', h)$ because they are not adjacent to $(g, h)$. If there exists any other vertex from the mentioned four, then such a vertex implies the existence of a vertex of label 0 in $G^h \cup H^g$, a contradiction. Hence we have only four vertices and $G \times H \cong K_2 \times K_2$. But in this case we have $\gamma_{tR}(G \times H) \leq 4$ as there exists a total Roman dominating function with $V_1 = V(G) \times V(H)$. This is the final contradiction and $\gamma_{tR}(G \times H) \neq 5$.

The implication ($\Rightarrow$) of item (ii) follows from (i) and the total Roman dominating function with $V_1 = V(K_2) \times V(K_2)$. For ($\Rightarrow$) of (ii) suppose that at least one of $G$ and $H$ contains more than three vertices. Hence $|V(G) \times V(H)| \geq 6$ and if all vertices have label 1, then $\gamma_{tR}(G \times H) \geq 6 > 4$. Otherwise, if $V_0 \neq \emptyset$, then also $V_2 \neq \emptyset$. Let $(g, h) \in V_2$ and let $(g', h') \in V_1 \cup V_2$ be a neighbor of $(g, h)$. If also $(g, h'), (g', h) \in V_1 \cup V_2$, then we have $\gamma_{tR}(G \times H) > 4$. On the other hand, if at least one of $(g, h')$ and $(g', h)$ has label 0, then there exists a vertex of label 2 different than $(g, h)$ and $(g', h')$, meaning that $\gamma_{tR}(G \times H) > 4$ again and (ii) is done.

For (iii) we start with ($\Rightarrow$). We know from (i) and (ii) that $\gamma_{tR}(G \times H) \geq 6$ whenever at least one of $G$ and $H$ contains more than two vertices, which is true in all three cases. Suppose first that each $G$ and $H$ have at least two universal vertices $g, g'$ and $h, h'$, respectively, and are of order at least three. If we set $V_2 = \{(g, h), (g', h')\}$, $V_1 = \{(g, h'), (g', h)\}$ and $V_0 = V(G) - (V_1 \cup V_2)$, then $f_1 = (V_0, V_1, V_2)$ is a total Roman dominating function with $\omega(f) = 6$. Assume now that one factor,
say \(H\), is \(K_2\) and that \(G\) contains at least three vertices together with a universal vertex \(g\). For \(V(H) = \{h, h'\}\) we define \(f_2 = (V'_0, V'_1, V'_2)\) by making \(V'_2 = \{(g, h), (g, h')\}, V'_1 = \{(g', h'), (g', h)\}\) and \(V'_0 = V(G) - (V_1 \cup V_2)\) for an arbitrary neighbor \(g'\) of \(g\) in \(G\). It is easy to check that \(f_2\) is a total Roman dominating function with \(\omega(f_2) = 6\). The third possibility is that both \(G\) and \(H\) are triangle centered graphs with central triangles \(g_1g_2g_3\) and \(h_1h_2h_3\), respectively. We define \(V''_2 = \{(g_1, h_1), (g_2, h_2), (g_3, h_3)\}\), \(V''_1 = \emptyset\) and \(V''_0 = V(G) - V_2\). We will show that \(f_3 = (V''_0, V''_1, V''_2)\) is a total Roman dominating function. First notice that \(V_2\) induces a triangle in \(G \times H\). Let \((g, h) \in V_0\). By the definition of the central triangle, \(g\) and \(h\) are adjacent to at least two vertices of \(\{g_1, g_2, g_3\}\) and \(\{h_1, h_2, h_3\}\), respectively. Hence, there exists \(i \in \{1, 2, 3\}\) such that \(g_i \in E(G)\) and \(h_i \in E(H)\), and \((g, h)\) is adjacent to \((g_i, h_i)\) in \(V_2\). Therefore, \(f\) is a total Roman dominating function on \(G \times H\) with \(\omega(f_3) = 6\). In all three cases we have \(\gamma_{tR}(G \times H) \leq 6\) and by (i) and (ii) the equality \(\gamma_{tR}(G \times H) = 6\) follows.

For the opposite implication (\(\Rightarrow\)) of (iii) we have \(\gamma_{tR}(G \times H) = 6\) and analyze the different possibilities for the cardinalities of \(V_1\) and \(V_2\) for a \(\gamma_{tR}(G \times H)\)-function \(f = (V_0, V_1, V_2)\). We start with \(|V_1| = 0\) and \(|V_2| = 3\) and let \((g_1, h_1), (g_2, h_2), (g_3, h_3) \in V_2\). As \(V_1 \cup V_2\) induces a graph without isolated vertices one vertex of the mentioned three, say \((g_2, h_2)\), must be adjacent to the other two. Hence \(g_1g_2, g_2g_3 \in E(G)\) and \(h_1h_2, h_2h_3 \in E(H)\). If \(g_1g_3 \notin E(G)\), then \((g_1, h_2)\) is a vertex of label 0 not adjacent to a vertex from \(V_2\). Similar, if \(h_1h_3 \notin E(H)\), then \((g_2, h_1)\) is a vertex of label 0 not adjacent to a vertex from \(V_2\). Hence \(g_1g_2g_3\) and \(h_1h_2h_3\) form a triangle in \(G\) and \(H\), respectively. Suppose that there exists \(g \in V(G)\) that is either adjacent to exactly one vertex of \(\{g_1, g_2, g_3\}\), say to \(g_1\), or to no vertex of \(\{g_1, g_2, g_3\}\). In both cases we obtain \((g, h_1)\) must has label 0, and is not adjacent to any vertex of \(V_1 \cup V_2\), which is not possible for a total Roman dominating function \(f\). Thus, every vertex \(g \in V(G)\) must be adjacent to at least two vertices from \(\{g_1, g_2, g_3\}\) and \(G\) is triangle centered. Similarly one shows that \(H\) is triangle centered and the third option follows.

We continue with \(|V_1| = 2\) and \(|V_2| = 2\). Let \((g, h)\) and \((g', h')\) be vertices of label 2. Assume first that \((g, h)\) and \((g', h')\) are adjacent. Hence, the vertices \((g, h')\) and \((g', h)\) are not adjacent to \((g, h)\) nor to \((g', h')\) and must have label 1. All the other vertices are in \(V_0\). Moreover, \(V_0 \neq \emptyset\) as the converse leads to a contradiction with \(f\) being a \(\gamma_{tR}(G \times H)\)-function. Every vertex \((g, x)\), \(x \in V(H) - \{h, h'\}\) has label 0 and is not adjacent to \((g, h)\). Therefore they must be adjacent to \((g', h')\), which means that \(h'\) is a universal vertex of \(H\). Similarly, every vertex \((g', x)\), \(x \in V(H) - \{h, h'\}\) has label 0 and is not adjacent to \((g', h')\). So they are adjacent to \((g, h)\), and \(h\) is a universal vertex of \(H\). By symmetric arguments, also \(g\) and \(g'\) are universal vertices of \(G\). Thus, both \(G\) and \(H\) have at least two universal vertices. If both have only two vertices, then we have a contradiction with (ii). Therefore we obtained the first possibility.

Let now \((g, h)\) and \((g', h')\) be nonadjacent. If they are not in the same \((G\)- or \(H\)-) layer, then \((g, h')\) and \((g', h)\) are not adjacent to \((g, h)\) nor to \((g', h')\) and must have label 1. All the other vertices must be in \(V_0\). But, this is a contradiction because \(V_1 \cup V_2\) induces four isolated vertices. Hence, \((g, h)\) and \((g', h')\) are in the same \(G\)- or \(H\)-layer, say in \(H^a\). So, \(g = g'\). If there exists different \(h_1, h_2 \in V(H) - \{h, h'\}\), then \((g, h_1), (g, h_2) \in V_1\), since there are no edges between vertices of \(H^a\). A contradiction again, due to no existing edges between vertices of \(V_1 \cup V_2\). If \(V(H) = 3\), say \(V(H) = \{h, h', h_1\}\), then \(f(g, h_1) = 1\) and the other vertex \((a, b)\) from \(V_1\) must be adjacent to all three vertices from \(H^a\). This is not possible as \((a, b)\) is contained in one of the
layers $G^h$, $G^{h'}$ or $G^{h_1}$. Again we have a vertex from $V_1 \cup V_2$ that is not adjacent to any other vertex of $V_1 \cup V_2$, a contradiction. So, $H$ contains only two vertices $h$ and $h'$, which are adjacent and therefore both universal vertices. If both vertices from $V_1$ belong to the same $G$-layer, say $G^h$, then $(g, h)$ is not adjacent to any vertex from $V_1 \cup V_2$, which is not possible. So, we may assume that $V_1 = \{(g_1, h), (g_2, h')\}$. Clearly $gg_1, gg_2 \in E(G)$, so that $V_1 \cup V_2$ induces a subgraph without isolated vertices. Also every vertex $(g_3, h) \in V_0$ must be adjacent to $(g, h')$, which means that $g g_3 \in E(G)$ and $g$ is an universal vertex of $G$. (Notice also that in the case when $g_1 = g_2$, there always exists $g_3 \in V(G) - \{g, g_1\}$, because otherwise we have a contradiction with $(ii)$.) This yields the middle case of $(iii)$.

To end with $(iii)$ let $|V_1| = 4$ and $|V_2| = 1$, where $V_2 = \{(g, h)\}$. Let $(g', h') \in V_1$ be a neighbor of $(g, h)$. Clearly all vertices from $G^h \cup H^g$ must be in $V_1 \cup V_2$, meaning that one of the factors is $K_2$ and the other contains three vertices, say $H \cong K_2$. Moreover, $g$ must be a universal vertex of $G$. So, either $G \cong C_3$ or $G \cong P_3$, which is the middle case of $(iii)$ and the proof of $(iii)$ is completed.

We continue with $(\Leftarrow)$ of $(iv)$. We may assume that $G$ has exactly one universal vertex $g$, and that $H$ is different from $K_2$ with a universal vertex $h$, and that at most one of $G$ and $H$ is triangle centered. Further, let $g'$ and $h'$ be arbitrary neighbors of $g$ in $G$ and of $h$ in $H$, respectively. By $(i)$, $(ii)$ and $(iii)$ we know that $\gamma_{tR}(G \times H) \geq 7$. If we set $V_2 = \{(g, h), (g, h'), (g', h)\}$, $V_1 = \{(g', h')\}$ and $V_0 = V(G \times H) - (V_1 \cup V_2)$, then $f = (V_0, V_1, V_2)$ is a total Roman dominating function with $\omega(f) = 7$. Hence, $\gamma_{tR}(G \times H) \leq 7$ and the equality follows.

For $(\Rightarrow)$ of $(iv)$, suppose that $\gamma_{tR}(G \times H) = 7$ and that $f = (V_0, V_1, V_2)$ is a $\gamma_{tR}(G \times H)$-function. First assume that $|V_1| = 1$ and $|V_2| = 3$, where $V_1 = \{(g_1, h_1)\}$ and $V_2 = \{(g_2, h_2), (g_3, h_3), (g_4, h_4)\}$. We may also assume that $(g_1, h_1)(g_2, h_2), (g_3, h_3)(g_4, h_4) \in E(G \times H)$ as $f$ is a $\gamma_{tR}(G \times H)$-function. Vertices $(g_3, h_3)$ and $(g_4, h_4)$ are not adjacent to $(g_3, h_3)$ nor to $(g_4, h_4)$. If $g_3 \neq g_2 \neq g_4$, then $(g_2, h_2)$ is adjacent to both $(g_3, h_4)$ and $(g_4, h_3)$ (even if one of them equals to $(g_1, h_1)$). As a consequence, we have $g_2 g_3, g_2 g_4 \in E(G)$ and $h_2 h_3, h_2 h_4 \in E(H)$. In other words, $g_2 g_3 g_4$ and $h_2 h_3 h_4$ form a triangle in $G$ and $H$, respectively. Let $g$ be an arbitrary vertex from $V(G) - \{g_2, g_3, g_4\}$ and let $h$ be an arbitrary vertex from $V(H) - \{h_2, h_3, h_4\}$. The vertex $(g, h)$ is adjacent to at least one vertex from $V_2$ (even if $(g, h) = (g_1, h_1)$). Let $(g_1, h_i)$ be a neighbor of $(g, h)$ for some $i \in \{2, 3, 4\}$. Clearly, $(g, h)$ and $(g, h_i)$ are not adjacent to $(g, h_i)$. Hence they must be adjacent to $(g_i, h_j)$ for some $j \in \{2, 3, 4\} - \{i\}$, meaning that $gg_j \in E(G)$ and $hh_j \in E(H)$. We see that both $G$ and $H$ are triangle centered graphs, and by $(iii)$ we have $\gamma_{tR}(G \times H) = 6$, a contradiction with $\gamma_{tR}(G \times H) = 7$.

So we can assume that either $g_2 = g_3$ or $g_2 = g_4$, say that $g_2 = g_3$. Moreover, also $h_2 = h_4$ as otherwise $(g_2, h_4)$ has no neighbor of label 2. If $h_2$ is not adjacent to some vertex $h \in V(H)$, then $(g_2, h)$ is not adjacent to a vertex of label 2, meaning that $h_2$ is a universal vertex of $H$. Similarly, we see that $g_2$ is a universal vertex of $G$. We have $\gamma_{tR}(G \times H) = 6$ by $(iii)$ when both $G$ and $H$ have (at least) two universal vertices, or one is $K_2$ and the other contains a universal vertex, a contradiction. Hence, one of $G$ or $H$ has at most one universal vertex and the other is not $K_2$ and we are done in this case.

The second possibility is that $|V_1| = 3$ and $|V_2| = 2$, where $V_1 = \{(g_1, h_1), (g_2, h_2), (g_3, h_3)\}$ and $V_2 = \{(g_4, h_4), (g_5, h_5)\}$. If $(g_4, h_4)$ and $(g_5, h_4)$ are adjacent, then $(g_4, h_5), (g_5, h_4) \in V_1$, say $(g_1, h_5) = (g_2, h_2)$ and $(g_5, h_4) = (g_3, h_3)$. Suppose that $g_1 \notin \{g_4, g_5\}$ and $h_1 \notin \{h_4, h_5\}$. All the
vertices of \( G^{h_4} - \{(g_4, h_4), (g_5, h_4)\} \) must be in \( V_0 \) and adjacent to \((g_5, h_5)\), meaning that \( g_5 \) is a universal vertex of \( G \). Similarly, all the vertices of \( G^{h_5} - \{(g_4, h_5), (g_5, h_5)\} \) must be in \( V_0 \) and adjacent to \((g_4, h_4)\), meaning that \( g_4 \) is a universal vertex of \( G \). This means that \( G \) is triangle centered with central triangle \( g_1 g_4 g_5 \). By symmetric arguments \( H \) is triangle centered with central triangle \( h_1 h_4 h_5 \). By (iii) we have \( \gamma_{tR}(G \times H) = 6 \), a contradiction. So, either \( h_1 \in \{h_4, h_5\} \) or \( g_1 \in \{g_4, g_5\} \), say \( h_1 = h_4 \). By the same arguments as above, we see that \( g_4 \) is a universal vertex of \( G \), and that \( h_4 \) and \( h_5 \) are universal vertices of \( H \). (Notice that \( g_1 \) is not adjacent to \( g_5 \), otherwise also \( g_5 \) is universal vertex, a contradiction with (iii).) If \( H \cong K_2 \), then we have \( \gamma_{tR}(G \times H) = 6 \) by (iii), a contradiction. Otherwise \( H \not\cong K_2 \) and we are done.

Now we can assume that \((g_4, h_4)\) and \((g_5, h_5)\) are not adjacent. If \( g_4 \neq g_5 \) and \( h_4 \neq h_5 \), then, as in the previous paragraph, we can choose the notation such that \((g_4, h_5) = (g_2, h_2)\) and that \((g_5, h_4) = (g_3, h_3)\). Moreover, \((g_1, h_1)\) must be adjacent to all other vertices from \( V_1 \cup V_2 \) in order to avoid isolated vertices of positive label. Vertices \((g_5, h_1)\) and \((g_1, h_4)\) are from \( V_0 \) and must have a neighbor in \( V_2 \). The only possibility is that \((g_5, h_1)\) is adjacent to \((g_4, h_4)\) and \((g_1, h_4)\) is adjacent to \((g_5, h_5)\). The mentioned edges imply that \( g_4 g_5 \in E(G) \) and \( h_4 h_5 \in E(H) \), a contradiction with the not adjacency of \((g_4, h_4)\) and \((g_5, h_5)\). It remains that \((g_4, h_4)\) and \((g_5, h_5)\) belong to the same layer, say \( H^{g_4} \), that is \( g_4 = g_5 \). Every vertex from \( H^{g_4} - \{(g_4, h_4), (g_4, h_5)\} \) is not adjacent to a vertex of label 2 and must poses label 1. We need also at least two vertices of label 1 outside of \( H^{g_4} \) to assure non isolated vertices in \( V_1 \cup V_2 \). This means that \( |V(H)| \leq 3 \). Every vertex from \( G^{h_4} - \{(g_4, h_4)\} \) must be adjacent to \((g_4, h_3)\) and \( g_4 \) is a universal vertex of \( G \). If \( H \cong K_2 \), then we have a contradiction with (iii). So either \( H \cong P_3 \) or \( H \cong C_3 \), meaning that also \( H \) has a universal vertex and the second possibility is done.

The last option is that \( |V_1| = 5 \) and \( |V_2| = 1 \), where \( V_2 = \{(g, h)\} \). Clearly all vertices from \( G^h \cup H^g \) must be in \( V_1 \cup V_2 \) and \( g \) and \( h \) must be universal vertices of \( G \) and \( H \), respectively. We either obtain a contradiction with (iii) (when one factor is \( K_2 \)) or obtain that \( G \cong H \cong K_{1,2} \) which yields the desired situation and the proof of (iv) is completed.

We conclude this proof with (v). We have \( \gamma_{tR}(G \times H) \geq 8 \) from assertions (i) – (iv). Let \( D_G = \{g, g'\} \) be a \( \gamma_t(G) \)-set and \( D_H = \{h, h'\} \) be a \( \gamma_t(H) \)-set. We set \( V_2 = D_G \times D_H \), \( V_1 = \emptyset \) and \( V_0 = V(G \times H) - V_2 \) and claim that \( f = (V_0, V_1, V_2) \) is a total Roman dominating function on \( G \times H \). Let \((g_1, h_1) \in V_0 \). Clearly, \( g_1 \) is adjacent to \( g \) or to \( g' \), say to \( g \), and \( h_1 \) is adjacent to \( h \) or to \( h' \), say to \( h \). Therefore \((g_1, h_1)\) is adjacent to \((g, h)\) and \( f \) is a total Roman dominating function on \( G \times H \). Hence \( \gamma_{tR}(G \times H) \leq 8 \) and equality follows. 

A wheel graph \( W_n \), \( n \geq 4 \), is a join of \( K_1 \) and \( C_{n-1} \) and a fan graph \( F_n \), \( n \geq 2 \), is a join of \( K_1 \) and \( P_{n-1} \). Clearly \( W_n \) and \( F_n \) have exactly one universal vertex when \( n > 4 \). In particular, \( W_n \) and \( F_n \) are triangle centered whenever \( n \in \{4, 5\} \). For a complete graph \( K_n \) and a maximum matching \( M \) of it, the graph \( K_n - M \), \( n \geq 5 \), is a triangle centered graph with a universal vertex whenever \( n \) is an odd number. From Theorem 10 we directly deduce the following results (among others).

**Corollary 11.** For integers \( n, m > 5, p \geq 1, q, s, t \geq 2, r > 2 \) and maximum matchings \( M \) and \( M' \) we have

(i) \( \gamma_{tR}(K_r \times K_s) = 6 \);
(ii) \( \gamma_{tR}(K_{1,s} \times K_{1,t}) = 7; \)

(iii) \( \gamma_{tR}(K_{p,q} \times K_{s,t}) = 8; \)

(iv) \( \gamma_{tR}(K_q \times K_{s,t}) = 8; \)

(v) \( \gamma_{tR}(K_r \times W_n) = 7; \)

(vi) \( \gamma_{tR}(K_r \times F_n) = 7; \)

(vii) \( \gamma_{tR}(W_n \times F_m) = 8; \)

(viii) \( \gamma_{tR}(W_n \times W_m) = 8; \)

(ix) \( \gamma_{tR}(F_n \times F_m) = 8; \)

(x) \( \gamma_{tR}((K_n - M) \times (K_m - M')) = 6. \)

With the help from Corollary 11, we can comment the sharpness for most of the bounds from Section 2. The upper bounds of Theorem 1, of Corollary 2 and of Remark 3 are sharp by (ii) of Corollary 11. The upper bound from Corollary 4 is sharp by (iii), (iv), (vii), (viii) and (ix) of Corollary 11. For \( p = q = s = t = 2 \) we have \( \gamma_{tR}(K_{2,2} \times K_{2,2}) = \gamma_{tR}(C_4 \times C_4) = 8 \) by (iii) of Corollary 11, and so the upper bound of Corollary 6 is sharp. The lower bound of Theorem 1 follows from \( \gamma_{tR}(P_4 \times P_4) = 8 = \rho(P_4)\gamma_{tR}(P_4) \) which holds by (v) of Theorem 10. By (iii) of Corollary 11, we show the sharpness of the bounds from Theorems 7 and 9 and Corollary 5. In conclusion, only the sharpness of the lower bound of Theorem 8 remains open.

We end this section with an alternative presentation with respect to Theorem 10, where we consider the number of vertices in \( V_1 \cup V_2 \) of a total Roman dominating function. For the minimum cardinality of \( V_1 \cup V_2 \), we need an additional condition that the cardinality of \( V_2 \) must be maximum to be able to characterize them.

**Theorem 12.** Let \( G \) and \( H \) be two graphs of order at least three and let \( f = (V_0, V_1, V_2) \) be a \( \gamma_{tR}(G \times H) \)-function with maximum \( V_2 \). The following assertions are equivalent.

(i) Graphs \( G \) and \( H \) are triangle centered.

(ii) \( \gamma_{tR}(G \times H) = 6. \)

(iii) \( |V_1 \cup V_2| = 3. \)

**Proof.** The direction \((i) \Rightarrow (ii)\) follows from (iii) of Theorem 10.

For the direction \((ii) \Rightarrow (iii)\), let \( \gamma_{tR}(G \times H) = 6 \) where \( f = (V_0, V_1, V_2) \) is a \( \gamma_{tR}(G \times H) \)-function with maximum cardinality of \( V_2 \). There exist vertices from \( G \times H \) in \( V_0 \) as there are at least nine vertices in \( G \times H \). Consequently \( V_2 \neq \emptyset \). Let \((g, h) \in V_2\) and let \((g', h')\) be a neighbor of \((g, h)\) with \( f(g', h') > 0 \). There exists at least one vertex \((x, y)\) from \((G^h \cup H^g) - \{(g, h)\}\) of label 0, because \( \gamma_{tR}(G \times H) = 6 \). Suppose that \((g'', h'')\) is a neighbor of \((x, y)\) of label 2. Assume first that \((g', h') = (g'', h'')\). The vertices \((g', h)\) and \((g, h')\) are not adjacent to \((g', h')\) nor to \((g, h)\). If they have label equal to 1, then all the other vertices have label 0 and every vertex is adjacent
to \((g, h)\) or to \((g', h')\). Let \(g_1\) and \(h_1\) be a third vertex of \(G\) and \(H\), respectively. Clearly, \((g_1, h_1)\) and \((g', h')\) are adjacent to \((g, h)\) and with this, we have \(g_1g_1 \in E(G)\) and \(h_1h_1 \in E(H)\). Similarly, \((g, h_1)\) and \((g_1, h)\) are adjacent to \((g', h')\), and with this we get \(g'g_1 \in E(G)\) and \(h'h_1 \in E(H)\). Let us define \(f' = (V_0', V_1', V_2')\) where \(V_0' = (V_0 \cup V_1) - \{(g_1, h_1)\}\), \(V_1' = \emptyset\) and \(V_2' = V_2 \cup \{(g_1, h_1)\}\). Clearly, \(f'\) is a total Roman dominating function with \(|V_2'| > |V_2|\), a contradiction with the choice of \(f\). Therefore, the label of \((g', h)\) and \((g', h')\) must be 0 and there exists a third vertex \((g_2, h_2)\) of label 2 that is adjacent to \((g', h)\) and \((g, h')\). From \(\gamma_{tr}(G \times H) = 6\) it follows that \(|V_1 \cup V_2| = 3\).

Next we assume that \((g', h') \neq (g'', h'')\). If also \(f(g', h') = 2\), then \(V_2 = \{(g, h), (g', h'), (g'', h'')\}\) and \(V_1 = \emptyset\) and we are done. So let \(f(g', h') = 1\). Because \(\gamma_{tr}(G \times H) = 6\) there exists a fourth vertex \((a, b)\) in \(V_1 \cup V_2\) with \(f(a, b) = 1\) and all other vertices are in \(V_0\). Vertex \((g'', h'')\) is not from \(G^h \cup H^g\), because \(V_2\) contains only \((g, h)\) and \((g'', h'')\) and we have at least three vertices in every \(G\)- or \(H\)-layer. Hence, \(g \neq g''\) and \(h \neq h''\). Vertices \((g'', h'')\) and \((g', h')\) are not adjacent to \((g, h)\) nor to \((g'', h'')\), and must therefore have label 1. This implies that \(\{(g, h'), (g'', h')\} = \{(g', h'), (a, b)\}\), which is a contradiction with \((g', h')\) being adjacent to \((g, h)\). Hence, \(|V_1 \cup V_2| = 3\) in all cases and this implication is done.

\(\left((iii) \Rightarrow (i)\right)\) Let \(|V_1 \cup V_2| = 3\) and let \((g_1, h_1), (g_2, h_2), (g_3, h_3) \in V_1 \cup V_2\). As \(V_1 \cup V_2\) induces a graph without isolated vertices, one vertex of these mentioned three, say \((g_2, h_2)\), must be adjacent to the other two. Thus, \(g_1g_2, g_2, g_1 \in E(G)\) and \(h_1h_2, h_2, h_3 \in E(H)\). If \(g_1g_3 \notin E(G)\), then \((g_1, h_2)\) is a vertex of label 0 not adjacent to a vertex from \(V_2\). Similarly, if \(h_1h_3 \notin E(H)\), then \((g_2, h_1)\) is a vertex of label 0 not adjacent to a vertex from \(V_2\). Hence \(g_1g_2g_3\) and \(h_1h_2h_3\) form a triangle in \(G\) and \(H\), respectively. Suppose that there exists \(g \in V(G)\) that is either adjacent to exactly one vertex of \(\{g_1, g_2, g_3\}\), say to \(g_1\), or to no vertex of \(\{g_1, g_2, g_3\}\). In both cases the vertex \((g, h_1)\) has label 0 and is not adjacent to any vertex of \(V_1 \cup V_2\), which is not possible for a total Roman dominating function \(f\). Hence, every vertex \(g \in V(G)\) must be adjacent to at least two vertices from \(\{g_1, g_2, g_3\}\) and \(G\) is triangle centered. Similarly, one shows that \(H\) is triangle centered. □

4 A general lower bound and its consequences on the direct product

The following lower bound for \(\gamma_{tr}(G)\) depends on the order of \(G\) and its maximum degree \(\Delta(G)\) as well as on a \(\gamma_{tr}(G)\)-function.

**Theorem 13.** If \(G\) is a graph with a \(\gamma_{tr}(G)\)-function \(f = (V_0, V_1, V_2)\), then \(\gamma_{tr}(G) \geq |V(G)| - (\Delta(G) - 2)|V_2|\) and \(\gamma_{tr}(G) \geq \frac{|V(G)| - |V_1|}{\Delta(G)}\). Moreover, if in addition \(|V(G)| = \Delta(G)|V_2| + |V_1|\), then the equality \(\gamma_{tr}(G) = |V(G)| - (\Delta(G) - 2)|V_2|\) holds.

**Proof.** Let \(f = (V_0, V_1, V_2)\) be a \(\gamma_{tr}(G)\)-function. Every vertex from \(V_2\) must have one neighbor in \(V_1 \cup V_2\). This means that every vertex from \(V_2\) can have at most \(\Delta(G) - 1\) neighbors in \(V_0\). With this we have

\[|V(G)| = |V_0| + |V_1| + |V_2| \leq (\Delta(G) - 1)|V_2| + |V_1| + |V_2|.\]  

From (2) we extract \(|V_2|\) and obtain the second inequality

\[|V_2| \geq \frac{|V(G)| - |V_1|}{\Delta(G)}.\]
Notice that from (2), it follows $|V_2|$ is maximum when $|V_1| = 0$. Now we return to (2), and add $0 = |V_2| - |V_2|$ on the right side to get

$$|V(G)| \leq (\Delta(G) - 2)|V_2| + |V_1| + 2|V_2| = (\Delta(G) - 2)|V_2| + \gamma_{tR}(G),$$  \hspace{1cm} (3)

that yields the first inequality. Notice that from the additional condition $|V(G)| = \Delta(G)|V_2| + |V_1|$ we get

$$|V_0| + |V_1| + |V_2| = |V(G)| = \Delta(G)|V_2| + |V_1|$$

and consequently $|V_0| = (\Delta(G) - 1)|V_2|$. This connection gives the equality in the lines (2) and (3) and the proof is completed. \hfill \Box

If we rewrite the Theorem 13 for the direct product $G \times H$, then we have the following.

**Corollary 14.** Let $G$ and $H$ be graphs. If $f = (V_0, V_1, V_2)$ is a $\gamma_{tR}(G \times H)$-function, then $\gamma_{tR}(G \times H) \geq |V(G)||V(H)| - (\Delta(G)\Delta(H) - 2)|V_2|$ and $|V_2| \geq \frac{|V(G)||V(H)| - |V_1|}{\Delta(G)\Delta(H)}$. Moreover, if in addition $|V(G)||V(H)| = \Delta(G)\Delta(H)|V_2| + |V_1|$, then the equality $\gamma_{tR}(G \times H) = |V(G)||V(H)| - (\Delta(G)\Delta(H) - 2)|V_2|$ holds.

The lower bound from Theorem 13 is better when $|V_2|$ is small as possible. Also, one cannot expect that the mentioned bound behave well when there exists a small number of vertices of maximum degree in $G$. From this point of view, one can expect that Theorem 13 works at its best for regular graphs. To see this, the following known remark is necessary.

**Remark 15.** [9] If $G$ is an efficient open domination graph with an efficient open dominating set $D$, then $D$ is a $\gamma_t(G)$-set.

**Theorem 16.** If $G$ is a regular efficient open domination graph, then $\gamma_{tR}(G) = 2\gamma_t(G)$.

*Proof.* Let $D$ be an efficient open dominating set of an $r$-regular graph $G$. By Remark 15 we have that $D$ is a $\gamma_t(G)$-set. Hence, $f = (V_0, V_1, V_2) = (V(G) - D, 0, D)$ is a total Roman dominating function on $G$ of weight $\omega(f) = 2\gamma_t(G)$ that clearly fulfills the condition $|V(G)| = \Delta(G)|V_2| + |V_1| = r|D|$. By Theorem 13 the result follows. \hfill \Box

The direct product $G \times H$ is an efficient open domination graph if and only if both factors $G$ and $H$ are efficient open domination graphs as shown in [1]. Moreover, $D_G \times D_H$ is an efficient open dominating set of $G \times H$ whenever $D_G$ and $D_H$ are efficient open dominating sets of $G$ and $H$, respectively. Hence we have the following result.

**Corollary 17.** If $G$ and $H$ are regular efficient open domination graphs, then $\gamma_{tR}(G \times H) = 2\gamma_t(G)\gamma_t(H)$.

The relaxation of Corollary 17 and Theorem 16 without the condition of regular graphs is not true anymore as shown by (ii) of Corollary 11. Clearly $K_{1,s}$ and $K_{1,t}$ are efficient open domination graphs that are not regular and we have $\gamma_{tR}(K_{1,s} \times K_{1,t}) = 7 \neq 8 = 2\gamma_t(K_{1,s})\gamma_t(K_{1,t})$.

A *prism* $P_G$ over a graph $G$ is a graph obtained from two copies of $G$ and a perfect matching between corresponding vertices of each copy (or the Cartesian product $G \square K_2$). All the prisms that are efficient open domination graphs are described in Theorem 4.3 from [9]. One 3-regular example is $P_{C_{3r}}$ and for them we have $\gamma_t(P_{C_{3r}}) = 2r$.

It is well known that a cycle $C_n$ is an efficient open domination graph if and only if $n \equiv 0 \, (\text{mod} \, 4)$ and the following result is clear by Corollary 17.
Corollary 18. If $m$ and $n$ are positive integers divisible by 4 and $t \geq 2$ and $r \geq 1$ are any integers, then

(i) $\gamma_{tR}(C_m \times C_n) = \frac{mn}{2}$;
(ii) $\gamma_{tR}(C_m \times K_{t,t}) = 2m$;
(iii) $\gamma_{tR}(C_m \times P_{C_{3r}}) = 2mr$;
(iv) $\gamma_{tR}(K_{t,t} \times P_{C_{3r}}) = 8r$.

References


